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# BALANCED AND QF-1 ALGEBRAS

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ABSTRACT. A ring $R$ is QF-1 if every faithful module has the double centralizer property. It is proved that a local finite dimensional algebra is QF-1 if and only if it is QF. From this it follows that an arbitrary finite dimensional algebra has the property that every homomorphic image is QF-1 if and only if every homomorphic image is QF.

Throughout the following all rings are associative and have identity, all modules are unitary and all algebras are finite dimensional over a field.

If $M$ is a module over a ring $R$, we write $\mathrm{End}(M)$ to represent its set of $R$-endomorphisms viewed as a ring of operators on the opposite side of $M$. Then $M$ is an $\mathrm{End}(M)$-$R$ bimodule, and calling its ring of $\mathrm{End}(M)$ endomorphisms $\mathrm{Bi\,End}(M)$, there is a natural ring homomorphism $R \to \mathrm{Bi\,End}(M)$ via $r \mapsto$ multiplication by $r$. If this ring homomorphism is surjective the module $M$ is said to be *balanced*, or to have the *double centralizer property*.

After proving, with C. J. Nesbitt, that every faithful module over a quasi-Frobenius (QF) algebra is balanced **[15]** (a fact now well known for QF rings (see, for example, **[2**, §59**]**)), R. M. Thrall gave an example of a ring over which every faithful module has the double centralizer property which is not QF **[16]**. He called the above rings QF-1 rings and posed the problem of characterizing them in terms of their ideal structure. Thrall's problem is still unsolved, even for algebras, though various partial results may be found in **[1]**, **[3]**, **[5]**, **[6]**, **[10]**.

Here, we offer a solution to a modification of Thrall's problem posed in **[1]** and **[3]**. A ring is *balanced* in case all of its modules (faithful or not) are balanced. It is our intention to prove that an algebra is balanced if and only if it is in fact a uniserial algebra in the sense of Koethe and Nakayama **[13]**. Thus we verify, for algebras, a recent conjecture of J. P. Jans **[8]** (cf. **[7**, Remark (d)**]**) and extend his theorem that a balanced

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algebra over an algebraically closed field is uniserial. In addition, we offer another partial solution to Thrall's problem: every primary decomposable QF-1 algebra is QF.

A ring is balanced if and only if each of its factor rings is QF-1. A ring is uniserial if and only if each of its factors is QF. Thus, uniserial rings are balanced, so that the converses to both theorems are well known to be true.

The problem is already reduced to the extent that we may assume that our algebra is local (i.e. a division ring modulo its radical). To see this, first recall that all the properties under consideration are preserved under finite ring direct products and summands (see **[5]** and **[13]**). Moreover, there are any number of ways to see that "uniserial" is a Morita invariant; and an unpublished theorem of K. Morita and H. Tachikawa **[12]** tells us that QF-1 and balanced are also. In particular, $R_n$ [$n\times n$ matrices over $R$] is balanced or QF-1 if and only if $R$ is. This fact, together with the fact that every balanced semiprimary ring is a product of matrix rings over local rings (**[7]**, **[8]**) tells us that we may assume our algebra is local.

In what follows we denote the Jacobson radical of a ring $R$ by $J$, and the right and left socles by Soc $R_R$ and Soc ${}_RR$. Since we will be working in an artinian ring, the right and left socles are the left and right annihilators of the radical, respectively. We note that if we limit ourselves to finitely generated modules the left and right QF-1 and balanced rings are the same. This is because if $R$ is a $K$ algebra, where $K$ is a field, then

$$\text{Bi End } M \approx \text{Bi End}(M^*)$$

where $M^*$ denotes the $K$ dual of the finitely generated module $M$.

We prove three lemmas, and then our theorems. The first of these requires the use of Lemma 11 of **[1]**. Let $A$ and $B$ be modules; define $T(A, B)=\sum \{\text{Im} f \mid f\in \text{Hom}_R(A, B)\}$. This lemma states

*If $A=\sum_{i\in I}\oplus A_i$ is a direct sum of right $R$ modules, and $\{r_i \mid i\in I\}\subset R$ then the map given by $\sum a_i\to\sum a_ir_i\in\sum\oplus A_i$ belongs to* Bi End $M$ *if and only if $T(A_i, A_j)(r_i-r_j)=0$ for all $i, j\in I$.*

We now prove our first lemma.

LEMMA A. *Let $R$ be local and right QF-*1, *and suppose $s$ and $t$ are nonzero elements of* Soc ${}_RR\cap$Soc $R_R$, *then $R/sR\approx R/tR$ as right $R$ modules. Equivalently, there are units $u$ and $v$ in $R$ such that $us=tv$.*

PROOF. The fact that $R/sR\approx R/tR$ if and only if there are units $u$ and $v$ such that $us=tv$ is straightforward.

Now, suppose $R/sR \not\approx R/tR$. Then, since $sR$ and $tR$ are minimal right ideals we must have $sR \cap tR = 0$. In particular, $R/sR \oplus R/tR$ is a faithful module. Let $A_1 = R/sR$, $A_2 = R/tR$, and $A_3 = R/(s+t)R$. Then, since $A_1$ is not isomorphic to $A_2$ we may, by interchanging $s$ and $t$ if necessary, assume that $A_2$ is not isomorphic to $A_3$. Consider the map defined by $(a_1, a_2, a_3)\gamma = (a_1 \cdot 0, a_2 \cdot s, a_3 \cdot 0)$. We claim that is is a Bi Endomorphism. To do this, we apply Lemma 11 [**1**]. Since $A_2$ is not isomorphic to $A_3$, we have $T(A_2, A_3) \subset J/(s+t)R$. If this were not true, we would be able to find a unit $u$, such that $ut = (s+t)v$, and since $(s+t)J = 0$, $v$ would necessarily be a unit and we would have $A_2 \approx A_3$. Similarly, $T(A_3, A_2) \subset J/tR$, $T(A_2, A_1) \subset J/sR$ and $T(A_1, A_2) \subset J/tR$. These are the only cases that involve a nonzero multiplication in the verification of the conditions of Lemma 11, and in these cases we will be multiplying by $\pm s$. Since $s \in \mathrm{Soc}\,{}_RR$, we have $Js = 0$, so the map defined is a Bi Endomorphism. Thus, this map must be given by right multiplication by an element $r \in R$. Now the map $\gamma$ is not zero since $(0, 1+tR, 0)\gamma = (0, s+tR, 0)$ and $sR \cap tR = 0$, thus $r \neq 0$. But since $r$ annihilates $A_1$ and $A_3$ we have $r = su$ and $r = (s+t)v$, where $u$ and $v$ are units. Thus, $su = (s+t)v$ or, $s(v-u) + tv = 0$, whence $tv = 0$ since the sum $sR + tR$ is direct, or $t = 0$, a contradiction, and the lemma is proved.

Note that this lemma is true under the sole hypothesis of right QF-1 and local.

To prove our next lemma we recall that if $I$ is a right ideal in a ring $R$, and $B = \{b \in R \mid bI \subset I\}$ then $\mathrm{End}(R/I)_R \approx B/I$ where the right side makes sense because $I$ is a two-sided ideal in $B$.

LEMMA B. *Suppose $R$ is a local right QF-1 ring with radical $J$ and that $0 \neq s \in \mathrm{Soc}\,{}_RR \cap \mathrm{Soc}\,R_R$. Let $B = \{b \in R \mid bs \in sR\}$. Then the radical of $B = J$, and $B/J$ is a division subring of $R/J$ such that the dimension of $R/J$ as a left vector space over $B/J$ is at most* 2.

PROOF. $Js = 0$ and inverses of units in $B$ belong to $B$, so $J \subset B$ and is clearly the radical of $B$. If $B = R$ we are done, so we may assume that there is a unit $u \in R$ such that $us$ is not zero and $usR \cap sR = 0$. Now if the conclusion fails we can find left $B$ modules $X$ and $Y$ such that $R/J = B/J \oplus X/J \oplus Y/J$, as left $B/J$ modules. Let $S = \mathrm{Soc}\,{}_RR \cap \mathrm{Soc}\,R_R$. Then $JS = 0$, and $us \in S$. Thus, $S/sR$ is not zero and $S$ and $sR$ are both left $B/J$ modules. Then, since $B/J$ is a division ring we can find a map $\delta_1 : R/J \to S/sR$ such that $(B/J \oplus X/J)\delta_1 = 0$ and $(Y/J)\delta_1 \neq 0$. Preceding $\delta_1$ by the canonical map $\delta_2 : R/sR \to R/J$ we obtain a nonzero $B$ map $\delta : R/sR \to R/sR$ with $((B+X)/sR)\delta = 0$. Thus, $\delta \in \mathrm{Bi\,End}\, R/sR$ so, by hypothesis, $\delta$ must be right multiplication by some nonzero $r \in R$. But then, $(B+X)r \subset sR$. In particular, $1 \cdot r \in sR$ and since $r \neq 0$, $rR = sR$. But then if $x \in X$ we have $xr \in sR$, or $x(rR)=rR$. From this we have $x(sR)=sR$, or $x \in B$, contrary to our choice of $x$.

Our final lemma requires the use of the field $K$. In this lemma at least, its only purpose is to permit the use of a dimension argument.

LEMMA C.[2] *If $R$ is a local QF-1 algebra over a field $K$ then $S=\mathrm{Soc}\, R_R \cap \mathrm{Soc}\, {}_R R$ is simple both as a left and a right $R$ module.*

PROOF. Let $S=\mathrm{Soc}\, R_R \cap \mathrm{Soc}\, {}_R R$, and let $0 \neq s \in S$. It is enough to prove that $sR$ is a two-sided ideal. For, if $0 \neq t \in S$ then $R/tR \approx R/sR$ by Lemma A so that there are units $u$ and $v$ such that $ut=sv$; but then $t=u^{-1}sv \in sR$, and counting dimension we have $tR=sR$ and $Rt=sR=Rs$.

Now let $s \in S$, and let $B$ be as in Lemma B and $B_1=\{r \in R \mid sr \in Rs\}$; $B_1$ is the opposite-sided counterpart of $B$. If either $B$ or $B_1$ is all of $R$ we are done, for if $B=R$ by the above remarks the conclusion holds, and if $B_1=R$, then $Rs$ is a two-sided ideal, and hence contains $sR$. But then, by dimension, $sR=Rs$ and $sR$ is a two-sided ideal. So, we may assume $B_1 \neq R$ and $B \neq R$. We will then show that $B_1 \cup B=R$ and arrive at a standard contradiction.

By Lemma B, we have that $R/J$ has left dimension equal to 2 over $B/J$, and since we are over a finite dimensional algebra and $B$ is a subalgebra, we have that $R/J$ has dimension 2 over $B/J$ on the right also. Now let $p \in R$ be such that $ps \notin sR$. Then $p \notin B$ so $R=pB+B$. Then $p$ is a unit, and $0 \neq ps+sp \in S$. By Lemma A, we can choose units $u$ and $v$ such that $us=(ps+sp)v$. Now write $u=pb_1+b_2$, $b_1, b_2 \in B$, and let $b_1 s=sr_1$ and $b_2 s=sr_2$. Then we have

$$(ps + sp)v = us, \qquad psv + spv = (pb_1 + b_2)s = psr_1 + sr_2$$

or, $0=ps(r_1-v)+s(r_2-pv)$.

Now, the sum $psR+sR$ is direct, so that $psr_1=psv$ and $sr_2=spv$. But since $p$ is a unit we have $sv=sr_1=b_1 s \neq 0$ so that $b_1$ is a unit and $sv^{-1}=b_1^{-1}s$.

Now, using the second equation and the calculations from the first, we have

$$sp = sr_2 v^{-1} = b_2 s v^{-1} = b_2 b_1^{-1} s \in Rs.$$

[2] (ADDED OCTOBER 13, 1971.) The two main theorems of this paper have been obtained independently by V. Dlab and C. M. Ringel. Their results are stated with the hypothesis "$R$ is artinian and finitely generated over its center $C$" in place of "$R$ is a finite dimensional algebra over a field $K$". Since a local ring $R$ satisfying their more general hypothesis has $R/J$ finite dimensional over the field $(C+J)/J$, the proofs of this lemma and the following theorems show that (as Dlab and Ringel state) if $R$ is such a ring then (1) if $R$ is right balanced then $R$ is uniserial; and (2) if $R$ is local and left and right QF-1 then $R$ is QF. (They also have shown that balanced rings are artinian, but need not be uniserial.)

Thus, if $p \notin B$ then $p \in B_1$. This says that $R=B\cup B_1$. They both contain $J$, and if we let $b \in B-B_1$ and $b_1 \in B_1-B$, where $b$ and $b_1$ are units, then $bb_1^{-1}$ is a unit not contained in either. (It is an old chestnut that a group cannot be a union of two proper subgroups.) This proves the lemma.

We are now ready to prove our theorems. Note that Theorem 1 follows from Theorem 2, but we have chosen to isolate it because there is a reasonable chance that the argument given here is extendable to artinian rings. See the remark after Theorem 2.

THEOREM 1. *Every balanced algebra is a uniserial algebra.*

PROOF. As discussed in our introductory remarks, we may assume that $R$ is local. Since Rad $R=J$ is nilpotent there is a largest integer $n$ such that $J^n \neq 0$. But then $J^n \subset \mathrm{Soc}\ {}_RR \cap \mathrm{Soc}\ R_R$ is left and right simple by Lemma C. In fact, applying Lemma C to each of the QF-1 algebras $R/J^{k+1}$ we see that $J^k/J^{k+1}$ is left and right simple for $k=1, \cdots, n$. Thus the lattices of one-sided ideals in $R$ are chains [**13**], so $R$ is uniserial.

THEOREM 2. *Every primary decomposable QF-*1 *algebra is QF.*

PROOF. Again we may assume the $K$ algebra $R$ is local. A local algebra is QF if either its left or right socle is simple. If, say, Soc ${}_RR$ is simple, then ${}_RR$ imbeds in $R_R^*$, the $K$ dual of $R_R$ which is injective and has the same $K$ dimension as ${}_RR$ (see [**2**, §60]), so ${}_RR$ is injective, since ${}_RR=R_R^*$. Now, according to Lemma C, if Soc $R_R \subset$ Soc ${}_RR$ the former is simple. So let us suppose that this is not the case. Then we have $x \in \mathrm{Soc}\ R_R$ such that $jx \neq 0$ for some $j \in J=\mathrm{Rad}\ R$. If $R/jxR \approx R/xR$ there are units $u$ and $v$ in $R$ such that $ujxv=x$. But then, for every $n$ we have $(uj)^n xv^n=x$. But this is impossible since $uj \in J$ and $J$ is nilpotent. Thus, $R/jxR$ is not isomorphic to $R/xR$ so we can apply [**1**, Lemma 11] (see Lemma A) to see that if $Jc=0$, then

$$(a + jxR, b + xR)\delta = (ac + jxR, b\cdot 0 + xR)$$

belongs to Bi End$(R/jxR \oplus R/xR)$. By hypothesis there is an $r \in R$ such that $R(c-r) \in jxR$ and $Rr \subset xR$. Now, if $r \neq 0$ by dimension $Rr=xR$ and $xR$ would be a two-sided ideal, contradicting the fact that $jx \notin xR$.

Thus, $r=0$ and for every $c \in \mathrm{Soc}\ {}_RR$, $c \in jxR$. Thus, Soc ${}_RR=jxR$ by dimensionality, and Soc ${}_RR$ is simple on the left since $jxR$ is simple on the right. This completes the proof.

REMARK. As noted in the introductory comments, Theorem 1 is really a consequence of Theorem 2. However, we chose to prove it independently in order to stress the point that our use of the finite dimensionality of $R$ over $K$ may be possible to avoid. Indeed, if one could find an element $S \in \mathrm{Soc}\ {}_RR \cap \mathrm{Soc}\ R_R$ for which $B = B_1$ in the proof of Theorem 1, one would be able to complete a proof that semiprimary balanced rings are uniserial.

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