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A NOTE ON PAIRED FIBRATIONS

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ABSTRACT. Consider pairs $(\mathscr{X}, \mathscr{A})$ where $\mathscr{X}=(X,p,B)$ and $\mathscr{A}=(A,p|A,B)$ are Hurewicz fibrations mapping onto B and $A\subset X$. It is proved that $(\mathscr{X}, \mathscr{A})$ is a cofibration if and only if $(\mathscr{X} \cup_f \mathscr{Y}, \mathscr{Y})$ is a strongly-paired fibration for each fibration $\mathscr{Y}=(Y,q,B)$ and fiber map $f:\mathscr{A}\to\mathscr{Y}$. It follows as a corollary that the notions of fiber homotopy equivalence and strong fiber homotopy equivalence [5] coincide for all Hurewicz fibrations. That $(\mathscr{X}, \mathscr{A})$ be "strongly-paired" requires more than that each lifting function for $\mathscr{A}$ be extendable to $\mathscr{X}$. This and other notions of pairing are studied.

1. **Introduction.** Throughout this note fibration will mean Hurewicz fibration with map onto the base space. All spaces are assumed to be Hausdorff. Suppose that $A\subset X$, that each of $\mathscr{X}=(X,p,B)$, $\mathscr{A}=(A,p|A,B)$, and $\mathscr{Y}=(Y,q,B)$ is a fibration and that $f:\mathscr{A}\to\mathscr{Y}$ is a fiber map. We let $\mathscr{X}\cup_f\mathscr{Y}=(X\cup_f Y, p\cup_f q, B)$, where $\cup_f$ denotes *adjunction* of spaces and maps. Is $\mathscr{X}\cup_f\mathscr{Y}$ a fibration? That is, do certain weak pushouts exist for Hurewicz fibrations? It is known [1] that the answer is yes provided that $(\mathscr{X}, \mathscr{A})$ is a cofibration *and* $\mathscr{A}$ is a subfibration of $\mathscr{X}$ in the sense that there is a lifting function for $\mathscr{A}$ which can be extended to a lifting function for $\mathscr{X}$.[2]

In this note we obtain a stronger result and its converse. Specifically, we define *strongly-paired fibration* and show that for fibrations $\mathscr{X}$ and $\mathscr{A}$ as above: *$(\mathscr{X}, \mathscr{A})$ is a cofibration if and only if $(\mathscr{X}\cup_f\mathscr{Y},\mathscr{Y})$ is a strongly-paired fibration whenever $\mathscr{Y}$ is a fibration and $f:\mathscr{A}\to\mathscr{Y}$ is a fiber map.*

We use this result to show that the notions of fiber homotopy equivalence and strong fiber homotopy equivalence coincide for all Hurewicz fibrations. Thus, we answer a question raised in [5].

2. **Notation and definitions.**

SUITABLE PATHS. A subset S of B^I is said to be *suitable* provided that it is closed under certain convenient "operations." Precisely, for $\omega\in B^I$

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and $s \in I$ we define ω_s and ω^s in B^I by

$$\omega^s(t) = \omega(s + (1 - s)t), \qquad \omega_s(t) = \omega(s(1 - t)).$$

Then the set S is suitable if and only if ω_s and ω^s are in S whenever $s \in I$ and $\omega \in S$.

PAIRING. For any $\mathscr{E}=(E, p, B)$ where p maps E onto B and any subspace S of B^I we let $\Omega(E, S)=\{(e, \omega) \in E\times S | p(e) = \omega(0)\}$. A *lifting* for $\Omega(E, S)$ is a map $\lambda:\Omega(E, S)\to E^I$ such that the "usual" lifting properties $\lambda(e, \omega)(0)=e$ and $p \circ \lambda(e, \omega) = \omega$ hold. It is trivially true that $\mathscr{E}$ is a fibration if and only if each $\Omega(E, S)$ has a lifting. Now, we say that $(\mathscr{X}, \mathscr{A})$, as above, is a *paired fibration* if and only if for each suitable S there is a lifting for $\Omega(A, S)$ which can be extended to a lifting for $\Omega(X, S)$. It is obvious that this is equivalent to saying $\mathscr{X}$ is a fibration and $\mathscr{A}$ is a subfibration. We say $(\mathscr{X}, \mathscr{A})$ is a *strongly-paired fibration* if and only if $\mathscr{A}$ is a fibration and, for each suitable S, *each* lifting for $\Omega(A, S)$ can be extended to a lifting for $\Omega(X, S)$.

It will be important to know that for a fixed base space B and fixed suitable S the operation of forming $\Omega(\ , S)$ commutes with that of forming adjunctions. Precisely, there is a natural homeomorphism between $\Omega(X\cup_f Y, S)$ and $\Omega(X, S)\cup_{\hat{f}} \Omega(Y, S)$, where $\hat{f}:\Omega(A, S)\to\Omega(Y, S)$ is defined by $\hat{f}(a, \omega)=(f(a), \omega)$.

GENERALIZED LIFTINGS. It is easy to show that if $\lambda:\Omega(E, S)\to E^I$ is a lifting and S is suitable, then there is a generalized lifting Λ which *extends* λ. That is, if we let $\Omega^*(E, S)=\{(e, \omega, s) \in E\times S\times I | \omega(s)=e\}$ then there is a map $\Lambda:\Omega^*(E, S)\to E^I$ such that $\Lambda(e, \omega, s)(s)=e$, $p\Lambda(e, \omega, s)=\omega$ and $\Lambda(e, \omega, 0)=\lambda(e, \omega)$. This fact is well known for $S=B^I$ **[2]** and the same proof will work for any suitable S.

COFIBRATIONS. With $\mathscr{X}$ and $\mathscr{A}$ as in the introduction we let $\mathscr{X}\times I$ denote $(X\times I, p \circ \pi_1, B)$ and $\mathscr{T}$ denote $(T, p \circ \pi_1|T, B)$ where $T=X\times \{0\} \cup A\times I$. We recall that $(\mathscr{X}, \mathscr{A})$ is a cofibration (**[3]**, **[5]**) if and only if there is a fiber map $\rho:\mathscr{X}\times I\to\mathscr{T}$ such that ρ is a retraction of $X\times I$ onto T.

3. Main results and proofs.

THEOREM 1. *Suppose that* $A\subset X$ *and that each of* $\mathscr{X}=(X, p, B)$ *and* $\mathscr{A}=(A, p|A, B)$ *is a fibration. Then, in order that* $(\mathscr{X}, \mathscr{A})$ *be a cofibration it is necessary and sufficient that* $(\mathscr{X}\cup_f \mathscr{Y}, \mathscr{Y})$ *be a strongly-paired fibration whenever* $\mathscr{Y}=(Y, q, B)$ *is a fibration and* $f:\mathscr{A}\to\mathscr{Y}$ *is a fiber map.*

COROLLARY 1. *If* $(\mathscr{X}, \mathscr{A})$ *is a cofibration of fibrations, then* $(\mathscr{X}, \mathscr{A})$ *is a strongly-paired fibration.*

COROLLARY 2. *If* $f:\mathscr{X}\to\mathscr{Y}$ *is a fiber map of fibrations over* B *and* $F:\mathscr{X}\times\{0\}\to\mathscr{Y}$ *is defined by* $F(x, 0)=f(x)$, *then the mapping cylinder* $\mathscr{X}\times I\cup_F \mathscr{Y}$ *is a fibration.*

THEOREM 2. *Any two fibrations over B are fiber homotopically equivalent if and only if they are strongly fiber homotopically equivalent.*

Before proving Theorem 1 we remark as follows upon the other results:

Corollary 1 follows from Theorem 1 when we observe that $(\mathcal{X}, \mathcal{A})$ is of the form $(\mathcal{X} \cup_f \mathcal{Y}, \mathcal{Y})$. Corollary 2 is a consequence of the fact that $(\mathcal{X} \times I, \mathcal{X} \times \{0\})$ is a cofibration.

Regarding Theorem 2, we recall that two fibrations $\mathcal{F}_0$ and $\mathcal{F}_1$ over B are said to be strongly fiber homotopically equivalent if and only if there is a fibration $\mathcal{C}$ over $B \times I$ such that $\mathcal{C}|B \times \{i\}$ is equivalent to $\mathcal{F}_i$ for $i=0$, 1. Theorems 4 and 10 of **[5]** imply that fiber homotopically equivalent $\mathcal{F}_i$ will be strongly fiber homotopically equivalent provided the mapping cylinder determined by a fiber homotopy equivalence $f: \mathcal{F}_0 \to \mathcal{F}_1$ is always a fibration. Corollary 2 says this is the case.

PROOF OF THEOREM 1.

SPECIAL CASE OF THE NECESSITY (COROLLARY 1). *If $(\mathcal{X}, A)$ is a cofibration, then $(\mathcal{X}, A)$ is strongly-paired.*

PROOF. Using previous notation we suppose that $\rho: \mathcal{X} \times I \to \mathcal{T}$ is a retraction map. Also suppose that $\phi: X \to I$ is a map such that $\phi(x)=0$ if and only if $x \in A$ and $\phi(x)<1$ implies $\pi_1\rho(x, 1) \in A$. It is known that such a ϕ can be defined, e.g., $\phi(x)=\sup_{t \in I} \{t-\pi_2\rho(x, t)\}$.

Now we let λ_A be a lifting for $\Omega(A, S)$ where S is suitable, Λ_A, a generalized lifting which extends λ_A, and Λ, any generalized lifting for X. That is $\Lambda_A: \Omega^*(A, S) \to A^I$ and $\Lambda: \Omega^*(X, S) \to X^I$ with

$$\Lambda_A(a, \omega, s)(s) = a, \qquad p\Lambda_A(a, \omega, s) = \omega,$$

$$\Lambda_A(a, \omega, 0) = \lambda_A(a, \omega), \qquad \Lambda(x, \omega, s)(s) = x, \quad \text{and} \quad p\Lambda(x, \omega, s) = \omega.$$

Next, define a map $k: \Omega(X, S) \to I$ by

$$k(x, \omega) = \max\{\phi(x), \phi(\Lambda(x, \omega, 0)(\phi(x)))\}.$$

Note that $0 \leqq \phi(x) \leqq k(x, \omega) \leqq 1$ and $k(x, \omega)=0$ if and only if $x \in A$. Also, $\pi_1\rho(\Lambda(x, \omega, 0)(\phi(x)), 1) \in A$ whenever $k(x, \omega)<1$. For convenience later we define j by $j(x, \omega)=\phi(x)+(1-k(x, \omega))$.

Finally, we define $\lambda: \Omega(X, S) \to X^I$ by

(1) $\lambda(x, \omega)(t)=\pi_1\rho(\Lambda(x, \omega, 0)(t), t/\phi(x))$ for $0 \leqq t < \phi(x)$,

(2) $\lambda(x, \omega)(\phi(x))=\pi_1\rho(\Lambda(x, \omega, 0)(t), 1)$,

(3) $\lambda(x, \omega)(t)=\Lambda_A(\lambda(x, \omega)(\phi(x)), \omega, \phi(x))(t)$ for $\phi(x)<t \leqq j(x, \omega)$, and

(4) $\lambda(x, \omega)(t)=\Lambda(\lambda(x, \omega)(j(x, \omega)), \omega, j(x, \omega))(t)$ for $j(x, \omega) \leqq t \leqq 1$.

To see that λ is well defined first note that formula (3) is only used when $\phi(x)<j(x, \omega)$ or, equivalently, when $k(x, \omega)<1$. In this case we have noted that $\pi_1\rho(\Lambda(x, \omega, 0)(\phi(x)), 1) \in A$. That is, from formula (2), $\lambda(x, \omega)(\phi(x)) \in A$. Thus $\Lambda_A(\Lambda(x, \omega)(\phi(x)), \omega, \phi(x))$ is defined as needed in (3). The other aspects of the fact that λ is well defined are easier to check and are left to the reader.

In order to prove that λ is continuous we consider closed sets C_1, C_2, and C_3 in $\Omega(X, S)\times I$ defined by

$$C_1 = \{((x, \omega), t) \mid 0 \leqq t \leqq \phi(x)\},$$
$$C_2 = \{((x, \omega), t) \mid \phi(x) \leqq t \leqq j(x, \omega)\},$$

and

$$C_3 = \{((x, \omega), t) \mid j(x, \omega) \leqq t \leqq 1\}.$$

Formulas (1) and (2) yield continuity on C_1; formulas (2) and (3) yield continuity on C_2; and formula (4) yields continuity on C_3.

NECESSITY PROOF. Let S be a suitable subset of B^I and $\lambda_Y:\Omega(Y, S)\to Y^I\subset(X\cup_f Y)^I$ be a lifting for $\mathscr{Y}$. Now recall that $\Omega(X\cup_f Y, S)$ is naturally homeomorphic to $\Omega(X, S)\cup_{\hat{f}}\Omega(Y, S)$. Thus it will suffice to find a map $\gamma:\Omega(X, S)\to(X\cup_f Y)^I$ such that $\gamma|\Omega(A, S)=\lambda_Y\circ\hat{f}$. For then we can set $\lambda = \gamma\cup_{\hat{f}}\lambda_Y$ to obtain a lifting for $\Omega(X\cup_f Y, S)$ which extends λ_Y.

Again we use the "cofibration maps" ρ and ϕ as in the proof of the special case.

Also, we use the result of the special case to choose a lifting λ_X for $\Omega(X, S)$ such that $\lambda_X|\Omega(A, S)$ is a lifting for $\Omega(A, S)$. Using these maps we define a map $m:\Omega(X, S)\to I$ by $m(x, \omega)=\sup_{t\in I}\{\phi(\lambda_X(x, \omega)(t))\}$. This m has the property that $m(x, \omega)=0$ if and only if $x\in A$. Also, $m(x, \omega)<1$ implies that $\pi_1\rho(\lambda_X(x, \omega)(t), 1)\in A$ for each $t\in I$.

Next we let $\nu:X\vee Y\to X\cup_f Y$ denote the natural map of the disjoint union of X and Y onto the adjunction space indicated. Finally, we let Λ_Y be a generalized lifting for $\Omega^*(Y, S)$ which extends λ_Y. Now the required γ can be defined by the following formulae:

(1) $\gamma(x, \omega)(t)=\nu\pi_1\rho(\lambda_X(x, \omega)(t), t/m(x, \omega))$ when $0\leqq t<m(x, \omega)$,

(2) $\gamma(x, \omega)(m(x, \omega))=\nu\pi_1\rho(\lambda_X(x, \omega)(m(x, \omega)), 1)$, and

(3) $\gamma(x, \omega)(t)=\nu(\Lambda_Y(f\pi_1\rho(\lambda_X(x, \omega)(m(x, \omega)), 1), \omega, m(x, \omega))(t))$ when $m(x, \omega)<t\leqq 1$.

Note that formula (3) is used only when $m(x, \omega)<1$ and in this case, as was noted above, $\pi_1\rho(\lambda_X(x, \omega)(m(x, \omega)), 1)\in A$. Consequently, γ is well defined in formula (3). It is easy to verify that γ is well defined on all of $\Omega(X, S)$. The continuity of γ can be checked by showing continuity on each of the closed subsets D_1 and D_2 of $\Omega(X, S)\times I$:

$$D_1 = \{((x, \omega), t) \mid 0 \leqq t \leqq m(x, \omega)\},$$
$$D_2 = \{((x, \omega), t) \mid m(x, \omega) \leqq t \leqq 1\}.$$

SUFFICIENCY PROOF. We let $Y=A\times I$, $q=p|A\circ\pi_1$ and $f(a)=(a, 0)$. Also let $S=\tilde{B}$ the set of constant paths in B and consider $\lambda_Y:\Omega(Y, S)\to Y^I\subset(X\cup_f Y)^I$ defined by $\lambda_Y((a, s), p^{\sim}(a))(t)=(a, s+(1-s)t)$. (For any point z, $\tilde{z}$ denotes the constant path at z.) The hypothesis guarantees a lifting $\lambda:\Omega(X\cup_f T, S)\to(X\cup_f Y)^I$ which extends λ_Y. We obtain $\rho:\mathscr{X}\times I\to\mathscr{T}$ by $\rho(x, t)=\lambda(x, p^{\sim}(x))(t)$. Here we identify X with its image in $X\cup_f Y$. We observe that ρ is a retraction since $\rho(x, 0)=\lambda(x, p^{\sim}(x))(0)=x$ and, if $a\in A$, $\rho(a, t)=\lambda(a, p^{\sim}(a))(t)=\lambda_Y((a, 0), p^{\sim}(a))(t)=(a, t)$. The lifting property of λ assures that ρ will preserve fibers. This completes the proof of the theorem.

4. **Some remarks, examples, and questions.** Returning to the definition of pairing we note that for $\mathscr{X}$ and $\mathscr{A}$ as above various types of pairing and strong-pairing could be defined. For *pairing* we would require that *some lifting* of a certain type for $\mathscr{A}$ *extend* to such a lifting for $\mathscr{X}$. For *strong-pairing* we would require that *each lifting* of a certain type for $\mathscr{A}$ *extend* to such a lifting for $\mathscr{X}$. We could vary the meaning of "a certain type." Two specific variations are indicated as follows:

(1) require the liftings only for $S=B^I$ rather than all suitable $S\subset B^I$ and obtain a definition of $(\mathscr{X}, \mathscr{A})$ being *B^I-paired* and *strongly-B^I-paired*,

(2) let ACHP stand for the absolute covering homotopy property and obtain definitions for $(\mathscr{X}, \mathscr{A})$ being *ACHP-paired* or *strongly-ACHP-paired.*

The exact formulation of these notions is left to the reader. We shall mention some relationships between these and other pairings. See Figure 1 where we suppose throughout that $\mathscr{X}$ and $\mathscr{A}$ are fibrations with $\mathscr{T}$ as above.

The arrows in Figure 1 indicate implications easily verified or following from the proof of Theorem 1. The equivalence of (2), (3) and (5) follows from the statement and proof of Theorem 1. That (1) implies (3) is easily proved using a regular lifting function (that is, one which lifts constant paths to constant paths), using $S=B^I$ rather than $S=\tilde{B}$ and proceeding as in the proof of Theorem 1.

The following examples show that certain implications do not hold:

(4)$\not\Rightarrow$(3). Consider X the comb space defined by $X=\{(x, y)\in\text{plane}\mid 0\leqq x\leqq 1 \text{ and } y=0\}\cup\{(x, y)\in\text{plane}\mid 0\leqq y\leqq 1, x=0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\}$, $A=\{(0, 1)\}$, $B=$a point. Here $(\mathscr{X}, \mathscr{A})$ is not a cofibration since A is a deformation retract of X without being a strong deformation retract. On the other hand, the fact that B is a point makes the strong-ACHP-pairing easy to verify.

(9)$\not\Rightarrow$(7). Consider $(\mathscr{T}, \mathscr{A}\times I)$ where $(\mathscr{X}, \mathscr{A})$ are as in the preceding example. Since $(\mathscr{X}, \mathscr{A})$ is not a cofibration, $(\mathscr{T}, \mathscr{A}\times I)$ is not strongly-paired, or equivalently (since B is a point), is not strongly-B^I-paired. However, $(\mathscr{T}, \mathscr{A}\times I)$ is paired because B is a point.

$$
\begin{tikzcd}[column sep=small]
\text{(1) } \mathscr{A} \text{ is regular and } (\mathscr{T}, \mathscr{A} \times I) \text{ is strongly-}B^I\text{-paired} \arrow[dr, Rightarrow] & & \\
\text{(2) } (\mathscr{T}, \mathscr{A} \times I) \text{ is strongly-paired} \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & \text{(3) } (\mathscr{X}, \mathscr{A}) \text{ is a cofibration} \arrow[d, Rightarrow] & \text{(4) } (\mathscr{X}, \mathscr{A}) \text{ is strongly-ACHP-paired} \arrow[dl, Rightarrow] \\
\text{(5) all } (\mathscr{X} \cup_f \mathscr{Y}, \mathscr{Y}) \text{ are strongly-paired} & \text{(6) } (\mathscr{X}, \mathscr{A}) \text{ is strongly-paired} \arrow[d, Rightarrow] & \\
 & \text{(7) } (\mathscr{X}, \mathscr{A}) \text{ is strongly-}B^I\text{-paired} \arrow[d, Rightarrow] & \\
\text{(8) } (\mathscr{X}, \mathscr{A}) \text{ is } B^I\text{-paired} \arrow[r, Leftrightarrow] & \text{(9) } (\mathscr{X}, \mathscr{A}) \text{ is paired} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \text{(10) } (\mathscr{X}, \mathscr{A}) \text{ is ACHP-paired} \\
 & \text{(11) } \mathscr{X} \text{ and } \mathscr{A} \text{ are both fibrations} &
\end{tikzcd}
$$

FIGURE 1

(11)$\not\Rightarrow$(9). Here we let $P(X)$ denote the nonregular path fibration given in [**4**]. That is, for an uncountable set J with the discrete topology let $X=\{x \in I^J \mid x(j)=0$ for at most one $j \in J\}$ and $P(X)=(X^I, p, X)$ with $p(\alpha)=\alpha(1)$. As before, we let $\tilde{X}$ denote the constant paths in X and $\tilde{P}(X)$ denote the trivial fibration $(\tilde{X}, p \mid \tilde{X}, X)$. It is easy to see that $(P(X), \tilde{P}(X))$ is not cofibration since then there would be a map $\phi: y^I \to X$ with $\phi^{-1}(0)=\tilde{X}$. Such a map would insure a regular lifting for any fibration over X [**4**]. The uncountability of J is crucial to the nonexistence of such a ϕ and, furthermore, this uncountability can be used in a usual fashion to show that $(P(X), \tilde{P}(X))$ is not even paired. The unique identity lifting for $\tilde{P}(X)$ cannot be extended to a lifting for $P(X)$. Details are left to the reader.

We raise the following questions: Does (7) imply (6)? Does (6) imply (4)? Does (3) imply (4)? While the three pairing notions (8), (9), and (10) are equivalent it seems that probably the analogous strong-pairing notions, that is, (7), (6) and (4), are not equivalent. The "natural" method for proving (7) implies (6) or (6) implies (4) would require a kind of "intermediary-extension property" which may not hold in general.

We conclude with the following remark on another failure for the strong-pairings:

The properties of being paired or being cofibered are both preserved by pullbacks. But, it appears that the strong pairings properties may fail to be so preserved. At least, here again, the "natural" method of proof will not work.

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