UNIQUENESS AND NORM CONVEXITY IN THE CAUCHY PROBLEM FOR EVOLUTION EQUATIONS WITH CONVOLUTION OPERATORS

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ABSTRACT. Uniqueness in the Cauchy problem is shown under suitable conditions for evolution equations of the form $u_t(x, t) - B(t, D_x)u(x, t)=0$, where B is a pseudo-differential operator of order $k \geq 0$ in the x variables. This is proved as a corollary to a norm convexity relation. In the process of showing this, an extension to Hölder's inequality is derived.

In recent work of Nirenberg-Treves [5], [6], investigations of local solvability of linear partial and pseudo-differential equations have led to the consideration of equations of the form

$$Lu(x, t) = D_t u(x, t) + iB(x, t, D_x)u(x, t),$$

where $(x, t) \in \Omega \times R$, $\Omega \subset R^n$, is a first order pseudo-differential operator in the x variables, whose principal symbol is real, and $D_j = -i\partial/\partial x_j$, $D_t = -i\partial/\partial t$. Also, the work of Calderón [1], [2], [3] has shown that considerations of uniqueness for the Cauchy problem for a system or a differential equation of arbitrary order often leads to a system of equations of this form.

In this article we allow B to have arbitrary nonnegative order k, and we consider $L = D_t + iB(t, D_x)$, where $D_x = (D_1, D_2, \cdots, D_n)$; i.e., B acts as a convolution operator with respect to the x variables. We remark that in [5], [6] it is shown how to reduce the more general $D_t + i(B + iA)$, where B and A both have real principal symbols, to this form in the case $k=1$ by using exponentials.

We show uniqueness in the Cauchy problem to $Lu=0$ for C^∞ solutions with compact support in the x variables for each fixed t, provided that the principal symbol b_k of B is semidefinite and behaves in a suitable fashion

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as a function of t. We state first a result on norm convexity:

THEOREM 1. *Suppose that*

$$(1) \qquad\qquad D_t u(x, t) + iB(t, D_x)u(x, t) = 0$$

on $\Omega \times [t_0, t_1]$, $\Omega \subset R^n$, $-\infty < t_0 < t_1 < \infty$, *such that the principal symbol* b_k *of* B *is a real positive semidefinite pseudo-differential operator of non-negative order* k *and that the symbol* b *of* B *satisfies* $b(t, \xi) = b_k(t, \xi) + g(t, \xi)$, *where* g *is the symbol of a pseudo-differential operator of degree* j, *and such that* j *is a nonnegative integer less than* k ($g \equiv 0$ *if* $k = 0$). *In addition, assume that, as a function of* t, *for each* ξ *fixed,* b_k *has zeros of finite order. Then, there exist positive constants* $\varepsilon = \varepsilon(t, B) \leqq \frac{1}{2}$ *and* $C = C(\varepsilon, B)$ *independent of* t *and* u *such that*

$$\|u(t)\| \leqq C[\|u(t_0)\|^{\varepsilon(t)} \|u(t_1)\|^{1-\varepsilon(t)} + \|u(t_0)\|^{1-\varepsilon(t)} \|u(t_1)\|^{\varepsilon(t)}]$$

for any t *in* (t_0, t_1), *where* $\| \ \| = \| \ \|_{L^2(R^n)}$. *Moreover, in any compact subset* $[t_0 + \varepsilon_1, t_1 - \varepsilon_1]$ *contained in* (t_0, t_1), *a uniform* ε *can be chosen independent of* t, *but depending on* ε_1.

REMARK. If B is negative semidefinite, let $t' = -t$ and the same theorem is valid.

COROLLARY 2. *Under the above hypotheses, if* $\|u\|$ *vanishes at either* $t = t_0$ *or* $t = t_1$, *then* $u \equiv 0$ *in* (t_0, t_1).

This result follows immediately by applying the theorem to any given t in (t_0, t_1).

A similar result can be shown for diagonal systems (which may arise from higher order equations) of the form

$$D_t u_j(x, t) + iB_j(t, D_x)u_j(x, t) = \sum_{l=1}^{m} G_{jl}(t, D_x)u_l(x, t), \quad j = 1, \cdots, m,$$

if each of the B_j satisfy the hypotheses of the theorem and the G_{jl} are of order $k - 1$. The method of proof is identical, but technical difficulties arise if the B_j's are not all the same. For details, see [7].

Calderón [1], [2], [3] proved that for general first order operators $B_j(x, t, D_x)$ with symbols b_j, one has local uniqueness in the Cauchy problem for systems if the roots of the characteristic equations $\tau_j - ib_j(x, t, \xi)$ (where $\xi = (\xi^1, \cdots, \xi^n)$ is the dual variable to x) satisfy the following criteria:

(a) real roots are simple,

(b) complex roots are at most double,

(c) roots do not coalesce, and

(d) imaginary parts of complex roots are bounded away from zero.

Our result in the case $k=1$ thus allows one to eliminate hypothesis (d) for this class of equations, and hence roots may wander from the real to the complex domain.

It is to be noted that our evolution operators all satisfy property (Ψ) of Nirenberg and Treves [5], and thus our operators are locally "solvable" (see [6]).

PROOF OF THEOREM 1.   We have $u_t - B(t, D_x)u = 0$. Then

$$\int_{R^n} e^{ix\cdot\xi}[\tilde{u}_t(\xi, t) - b(t, \xi)\tilde{u}(\xi, t)]\, d\xi = 0,$$

where "~" denotes the Fourier transform in the x-variables. The Fourier inversion formula then yields the ordinary differential equation $\tilde{u}_t(\xi, t) - b_k(t, \xi)\tilde{u}(\xi, t) = g(t, \xi)\tilde{u}(\xi, t)$. Set $\beta(t, \xi) = \int_{t_0}^t b_k(s, \xi)\, ds$, so that $\beta(t, \xi)$ is homogeneous of degree k in $|\xi|$ for $|\xi| > 1$ and is the symbol of a pseudo-differential operator of order k for each fixed t in $[t_0, t_1]$. Moreover, $\beta(t_0, \xi) = 0$ and $\beta(t, \xi)$ is real and monotone increasing in t for ξ fixed, as $b_k \geqq 0$ and the zeros of $b_k(t)$ are of finite order.

Then $(d/dt)(e^{-\beta(t,\xi)}\tilde{u}(\xi, t)) = e^{-\beta(t,\xi)}g(t, \xi)\tilde{u}(\xi, t)$, which has as solutions

(2) $$e^{-\beta(t,\xi)}\tilde{u}(\xi, t) = \tilde{u}(\xi, t_0) + \int_{t_0}^t e^{-\beta(s,\xi)}g(s, \xi)\tilde{u}(\xi, s)\, ds$$

and

(3) $$e^{-\beta(t,\xi)}\tilde{u}(\xi, t) = e^{-\beta(t_1,\xi)}\tilde{u}(\xi, t_1) - \int_t^{t_1} e^{-\beta(s,\xi)}g(s, \xi)\tilde{u}(\xi, s)\, ds.$$

We are seeking a technique that will yield uniqueness for both the forward and backward problems simultaneously, so we want to combine formulas (2) and (3) in such a way as to eliminate the exponential terms.

First, however, we must take care of the integral expressions in (2) and (3). We set

$$w(\xi, t) = |e^{-\beta(t,\xi)}\tilde{u}(\xi, t)| = e^{-\beta(t,\xi)}|\tilde{u}(\xi, t)| \geqq 0.$$

Then (2) and (3) become respectively

$$w(\xi, t) = w(\xi, t_0) + \int_{t_0}^t g(s, \xi)w(\xi, s)\, ds \quad \text{for } t \in [t_0, t_1]$$

and

$$w(\xi, t) = w(\xi, t_1) - \int_t^{t_1} g(s, \xi)w(\xi, s)\, ds \quad \text{for } t \in [t_0, t_1].$$

On $[t_0, t_1]$, $g(s, \xi)$ is bounded by $C_0|\xi|^j$ for $|\xi| > 1$, where C_0 is a positive constant independent of t and ξ. Hence

(4) $$w(\xi, t) \leqq w(\xi, t_0) + C_0|\xi|^j \int_{t_0}^t w(\xi, s)\, ds$$

and

(5) $$w(\xi, t) \leqq w(\xi, t_1) + C_0 |\xi|^j \int_t^{t_1} w(\xi, s) \, ds.$$

Set

$$\gamma = \gamma(\xi) = 2C_0 |\xi|^j > 0$$

and

$$M_0(\xi) = \max_{t \in [t_0, t_1]} \exp[-\gamma(t - t_0)] w(\xi, t),$$

$$M_1(\xi) = \max_{t \in [t_0, t_1]} \exp[-\gamma(t_1 - t)] w(\xi, t).$$

We claim

(6) $$M_0(\xi) \leqq 2w(\xi, t_0) \quad \text{and} \quad M_1(\xi) \leqq 2w(\xi, t_1).$$

To show the first part, we suppose M_0 occurs at $t = \bar{t}$, $t_0 < \bar{t} \leqq t_1$. Then a straightforward computation shows that

$$M_0(\xi) \leqq 2 \exp[-\gamma(\bar{t} - t_0)] w(\xi, t_0) \leqq 2w(\xi, t_0) \quad \text{by using (4).}$$

Similarly, using (5), if M_1 occurs at $t = t'$, $t_0 \leqq t' < t_1$, one sees that

$$M_1(\xi) \leqq 2 \exp[-\gamma(t_1 - t')] w(\xi, t_1) \leqq 2w(\xi, t_1).$$

Formulas (6) state that

$$\exp[-\gamma(t - t_0)] w(\xi, t) \leqq 2w(\xi, t_0)$$

and

$$\exp[-\gamma(t_1 - t)] w(\xi, t) \leqq 2w(\xi, t_1);$$

and thus

(7) $$e^{-\beta(t,\xi)} |\tilde{u}(\xi, t)| \leqq 2 \exp[2C_0 |\xi|^j (t - t_0)] |\tilde{u}(\xi, t_0)|$$

and

(8) $$e^{-\beta(t,\xi)} |\tilde{u}(\xi, t)| \leqq 2 \exp[2C_0 |\xi|^j (t_1 - t)] \exp[-\beta(t_1, \xi)] |\tilde{u}(\xi, t_1)|$$

for $-\infty < t_0 \leqq t \leqq t_1 < \infty$.

Next we use the fact that $|\xi|^j \leqq \delta |\xi|^k + C_\delta$ for $|\xi| > 1$ and any $\delta > 0$, where $C_\delta = \delta^{-j/(k-j)}$. Then (7) and (8) become

(9) $$|\tilde{u}(\xi, t)| \leqq 2 \exp[2C_0(t - t_0)(\delta |\xi|^k + C_\delta) + \beta(t, \xi)] |\tilde{u}(\xi, t_0)|$$

and

(10) $$|\tilde{u}(\xi, t)| \leqq 2 \exp[2C_0(t_1 - t)(\delta |\xi|^k + C_\delta) + \beta(t, \xi) - \beta(t_1, \xi)] |u(\xi, t_1)|,$$
$$\text{for any } \delta > 0 \text{ and } t \in [t_0, t_1].$$

We restrict our consideration to those t satisfying $t_0+\varepsilon_1=\tau_0\leqq t\leqq\tau_1=t_1-\varepsilon_1$. Since $b_k\geqq0$, $\beta_0(t,\xi)=|\xi|^{-k}\beta(t,\xi)$ is the principal symbol of a peudo-differential operator of order zero and is monotone increasing in t for each ξ fixed ($|\xi|>1$). Then we set $\eta(t,\xi)=[\beta_0(t,\xi)+2C_0\delta(t-t_0)]|\xi|^k$, which is homogeneous of degree k in ξ for $|\xi|>1$, and choose

$$\delta < \frac{\varepsilon'}{2C_0}\min_{t\in[\tau_0,\tau_1]}\inf_{|\xi|>1}\frac{\beta_0(t,\xi)}{t-t_0}, \quad \text{where } \varepsilon'\leqq\frac{\beta_0(t,\xi)}{\beta_0(t_1,\xi)}\leqq 1-\varepsilon'$$

in $[\tau_0,\tau_1]$. This can be done because β_0 is the principal symbol of a pseudo-differential operator of order zero.

Then on $[\tau_0,\tau_1]$ and for $|\xi|>1$,

$$\frac{\beta_0(t,\xi)}{\beta_0(t_1,\xi)+2C_0\delta(t_1-t_0)} > \frac{\beta_0(t,\xi)}{\beta_0(t_1,\xi)}\frac{1}{1+\varepsilon'} = \frac{1}{1+\varepsilon'}\frac{\beta(t,\xi)}{\beta(t_1,\xi)}$$

and

$$\frac{\beta_0(t,\xi)+2C_0\delta(t-t_0)}{\beta_0(t_1,\xi)} > \frac{\beta_0(t,\xi)}{\beta_0(t_1,\xi)}(1+\varepsilon') = (1+\varepsilon')\frac{\beta(t,\xi)}{\beta(t_1,\xi)}.$$

Thus,

$$0 < \varepsilon'/(1+\varepsilon') < \eta(t,\xi)/\eta(t_1,\xi) < (1+\varepsilon')(1-\varepsilon') < 1$$

for $|\xi|>1$ and $\tau_0\leqq t\leqq\tau_1$. Formulas (9) and (10) become

(11) $$|\tilde{u}(\xi,t)| \leqq N_0 e^{\eta(t,\xi)}|\tilde{u}(\xi,t_0)|$$

and

(12) $$|\tilde{u}(\xi,t)| \leqq N_0\exp[\eta(t,\xi)-\eta(t_1,\xi)]|\tilde{u}(\xi,t_1)|$$

for $|\xi|>1$ and $\tau_0\leqq t\leqq\tau_1$, where $N_0=2\exp[2C_0(t_1-t_0)C_\delta]$ and $\eta(t,\xi)$ is the principal symbol of a pseudo-differential operator of order k.

Next we set $\alpha(t,\xi)=\eta(t,\xi)/\eta(t_1,\xi)$ for $|\xi|>1$, raise (11) and (12) to the $1-\alpha(t,\xi)$ and $\alpha(t,\xi)$ powers respectively, and multiply the two formulas together. Hence

(13) $$|\tilde{u}(\xi,t)| \leqq N_0|\tilde{u}(\xi,t_0)|^{1-\alpha(t,\xi)}|\tilde{u}(\xi,t_1)|^{\alpha(t,\xi)} \quad \text{for } |\xi| > 1.$$

Note that $0\leqq\alpha(t,\xi)\nearrow1$ for ξ fixed, since $\eta(t,\xi)$ is monotone increasing in t. If α were independent of ξ we could square the terms in (13), integrate for $|\xi|>1$ and get the relationship we want by using Hölder's inequality. However, by exploiting the facts that α is homogeneous of degree k in $|\xi|$ for $|\xi|>1$ and that there exists $\varepsilon=\varepsilon(t)>0$ such that $\varepsilon\leqq\alpha(t,\xi)\leqq1-\varepsilon$ for t fixed, we can use an extension of Hölder's inequality (Lemma 3 below).

This tells us that, for any fixed t in (t_0, t_1),

$$\int_{|\xi|>1} |\tilde{u}(\xi, t)|^2 \, d\xi \leq \left(\int_{|\xi|>1} |\tilde{u}(\xi, t_0)|^2 \, d\xi \right)^{1-\varepsilon} \left(\int_{|\xi|>1} |\tilde{u}(\xi, t_1)|^2 \, d\xi \right)^{\varepsilon}$$
$$+ \left(\int_{|\xi|>1} |\tilde{u}(\xi, t_0)|^2 \, d\xi \right)^{\varepsilon} \left(\int_{|\xi|<1} |\tilde{u}(\xi, t_1)|^2 \, d\xi \right)^{1-\varepsilon}$$
$$\leq \|\tilde{u}(\cdot, t_0)\|^{2(1-\varepsilon)} \|\tilde{u}(\cdot, t_1)\|^{2\varepsilon} + \|\tilde{u}(\cdot, t_0)\|^{2\varepsilon} \|\tilde{u}(\cdot, t_1)\|^{2(1-\varepsilon)}.$$

We may also write

$$(14) \quad \int_{|\xi| \leq 1} |\tilde{u}(\xi, t)|^2 \, d\xi$$
$$\leq C \int_{|\xi| \leq 1} |e^{\eta(t, \xi)} \tilde{u}(\xi, t_0)|^{2(1-\varepsilon)} |e^{\eta(t, \xi) - \eta(t_1, \xi)} \tilde{u}(\xi, t_1)|^{2\varepsilon} \, d\xi,$$

since g is bounded on $\{|\xi| \leq 1\}$. In the compact set $\{|\xi| \leq 1\}$ the exponentials are bounded, so the integrals make sense. Then

$$\int_{|\xi| \leq 1} |\tilde{u}(\xi, t)|^2 \, d\xi \leq C(\eta, \varepsilon) \int_{|\xi| \leq 1} |\tilde{u}(\xi, t_0)|^{2(1-\varepsilon)} |\tilde{u}(\xi, t_1)|^{2\varepsilon} \, d\xi$$
$$(15) \quad \leq C(\eta, \varepsilon_1) \left(\int_{|\xi| \leq 1} |\tilde{u}(\xi, t_0)|^2 \, d\xi \right)^{1-\varepsilon} \left(\int_{|\xi| \leq 1} |\tilde{u}(\xi, t_1)|^2 \, d\xi \right)^{\varepsilon}$$
$$\leq C(\eta, \varepsilon_1) \|\tilde{u}(\cdot, t_0)\|^{2(1-\varepsilon)} \|\tilde{u}(\cdot, t_1)\|^{2\varepsilon},$$

as we can use the standard Hölder inequality here, with $p = 1/(1-\varepsilon)$, $q = 1/\varepsilon$. We add (14) and (15) to get

$$\|\tilde{u}(\cdot, t)\|^2 \leq (C(\eta, \varepsilon) + 1)[\|\tilde{u}(\cdot, t_0)\|^{2(1-\varepsilon)} \|\tilde{u}(\cdot, t_1)\|^{2\varepsilon}$$
$$+ \|\tilde{u}(\cdot, t_0)\|^{2\varepsilon} \|\tilde{u}(\cdot, t_1)\|^{2(1-\varepsilon)}].$$

Set $C = C(\eta, \varepsilon) + 1$. Then, by applying Parseval's theorem, we obtain the desired inequality.

If $[t_0 + \varepsilon_1, t_1 - \varepsilon_1]$ is any compact subset of (t_0, t_1), then $\varepsilon = \varepsilon(\varepsilon_1, \eta)$ can be found so that $\varepsilon \leq \alpha(t, \xi) \leq 1 - \varepsilon$ on $[t_0 + \varepsilon_1, t_1 - \varepsilon_1]$, and the constant in the inequality will become $\max_{t \in [t_0 + \varepsilon_1, t_1 - \varepsilon_1]} (C(\eta, \varepsilon) + 1)$ in this case.

$$\text{Q.E.D.}$$

It remains to state and prove Lemma 3, which is of independent interest since it is an extension of Hölder's inequality.

LEMMA 3. *Let f, g be in $L^2(y)$ and $0 < \varepsilon \leq \gamma(y) \leq 1 - \varepsilon < 1$. Then*

$$\int |f|^{2\gamma(y)} |g|^{2(1-\gamma(y))} \, dy \leq \|f\|^{2(1-\varepsilon)} \|g\|^{2\varepsilon} + \|f\|^{2\varepsilon} \|g\|^{2(1-\varepsilon)},$$

where $\| \quad \| = \| \quad \|_{L^2(y)}$.

Note that if $\gamma(y) \equiv \gamma_0 =$ constant, then the ordinary Hölder inequality implies that

$$\int |f|^{2\gamma_0} |g|^{2(1-\gamma_0)} \, dy \leq \|f\|^{2\gamma_0} \|g\|^{2(1-\gamma_0)},$$

so that by allowing γ to vary, the bound becomes the sum of the bounds in Hölder's inequality for $\gamma_0 = \varepsilon$ and $1 - \varepsilon$ respectively.

PROOF OF LEMMA 3.   The proof hinges on the pointwise inequality

(16) $\delta^{\gamma(y)} \eta^{1-\gamma(y)} \leq \max(\delta^\varepsilon \eta^{1-\varepsilon}, \delta^{1-\varepsilon} \eta^\varepsilon)$ for γ as above,

and any nonnegative constants δ and η as above. For, once we have this inequality, then $\delta^{\gamma(y)} \eta^{1-\gamma(y)} \leq \delta^\varepsilon \eta^{1-\varepsilon} + \delta^{1-\varepsilon} \eta^\varepsilon$; so that if we set $f(y) = \delta$, $g(y) = \eta$ and integrate over all y, the ordinary Hölder inequality with $p = 1/\varepsilon$, $q = 1/(1-\varepsilon)$ yields the result. Thus, we must only verify (16). Suppose $\delta^\varepsilon \eta^{1-\varepsilon} \geq \delta^{1-\varepsilon} \eta^\varepsilon$. Then $\eta^{1-2\varepsilon} \geq \delta^{1-2\varepsilon}$; i.e., $(\eta/\delta)^{1-2\varepsilon} \geq 1$, and thus $\eta/\delta \geq 1$. (If $\varepsilon = \frac{1}{2}$, this procedure is unnecessary.) So

$$\delta^{\gamma(y)} \eta^{1-\gamma(y)} \leq \delta^\varepsilon \eta^{1-\varepsilon} \delta^{\gamma(y)-\varepsilon} \eta^{\varepsilon-\gamma(y)} = \delta^\varepsilon \eta^{1-\varepsilon} (\delta/\eta)^{\gamma(y)-\varepsilon}$$

$$< \delta^\varepsilon \eta^{1-\varepsilon} = \max(\delta^\varepsilon \eta^{1-\varepsilon}, \delta^{1-\varepsilon} \eta^\varepsilon).$$

The proof for $\delta^{1-\varepsilon} \eta^\varepsilon \geq \delta^\varepsilon \eta^{1-\varepsilon}$ is similar.   Q.E.D.

REMARK.   If $b = b_k$ and is independent of t, then

$$\alpha(t, \xi) = \frac{b_k(\xi)(t - t_0)}{b_k(\xi)(t_1 - t_0)} = \frac{t - t_0}{t_1 - t_0}$$

is independent of ξ. Hence the result in this case is

$$\|u(t)\| \leq C(\|u(t_0)\|^{(t_1-t)/(t_1-t_0)} \|u(t_1)\|^{(t-t_0)/(t_1-t_0)}$$

$$+ \|u(t_0)\|^{(t-t_0)/(t_1-t_0)} \|u(t_1)\|^{(t_1-t)/(t_1-t_0)}).$$

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