GENERALIZATIONS OF RESULTS OF AGMON AND NIRENBERG TO THE CAUCHY PROBLEM FOR EVOLUTION EQUATIONS

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ABSTRACT. Uniqueness and norm convexity for evolution equations are investigated by using tools developed by Agmon and Nirenberg.

We continue here our study of uniqueness for the Cauchy problem for evolution equations begun in [7]. We investigate

$$Lu(x, t) = D_t u(x, t) + i(B(x, t, D_x) + iA(x, t, D_x))u(x, t)$$

$$= -i[u_t(x, t) - (B(x, t, D_x) + iA(x, t, D_x))u(x, t)],$$

where B and A are pseudo-differential operators in the x-variables for $(x, t) \in R^n \times R^1$ whose principal symbols are real, $D_{x_j} = -i\partial/\partial x_j$, and $D_t = -i\partial/\partial t$. In [7], we examined the equation $Lu=0$ for those B that were convolution operators in the x variables, i.e. B independent of t. Here we will allow B to depend on x, but restrict it in another fashion. In addition to proving uniqueness and a (sharper) norm convexity result as in the previous investigation, we are able to deduce a formula on decay of solutions.

Our work is based on some results of Agmon and Nirenberg ([1], [2], [3]), which in turn are related to the Hadamard Three-Circle Theorem, and we will only indicate how the proofs follow from their work. We use pseudo-differential operators; for their properties see [5].

Our results are stated for single equations. They are also valid for diagonal systems whose component equations satisfy the hypotheses. For details see [6].

THEOREM 1.   *Suppose that $u(x, t) \in C^\infty(\Omega \times [t_0, t_1))$ with compact support in Ω for each fixed t, where $\Omega \subset R^n$, and satisfies*

$$(1) \quad u_t(x, t) - (B(x, t, D_x) + iA(x, t, D_x))u(x, t) = f, \quad \|f\| \leqq \Phi(t)\|u\|,$$

where B and A are first order pseudo-differential operators whose principal symbols are real, Φ is nonnegative, bounded, continuous, and integrable in t and $\| \quad \| = \| \quad \|_{L^2(R^n)}$. Assume also that one of the following is valid for $t \in [t_0, t_1)$:

   (a) *B is elliptic of order one; i.e. $\|\Lambda v\| \leqq C(t)(\|Bv\| + \|v\|)$ for all v in $C_0^\infty(\Omega)$, where $\Lambda = (1 - \Delta)^{1/2}$, $\Delta = Laplacian$;*
   (b) *B is of order zero instead of order one;*
   (c) *$\|B_t v\| \leqq C(t)\|Bv\|$ for all v in $C_0(\Omega)$, where $B_t = dB/dt$; or*
   (d) *the principal symbol g of B_t is nonnegative.*

Then if $\|u(\bar t)\| = 0$ for $\bar t \in [t_0, t_1)$, u vanishes identically in $\Omega \times [t_0, t_1)$. Also, if $\|u(t)\| \neq 0$ for all $t \in [t_0, t_1)$, then $\|u(t)\| \geqq \|u(t_0)\| \exp[-\mu(t-t_0)]$ for $t_1 > t \geqq t_0 + 1$, where μ is a constant independent of t. Finally, if $u(t) \neq 0$ for all $t \in [\tau_0, \tau_1)$, where $t_0 \leqq \tau_0 \leqq t < \tau_1 \leqq t_1$, then

$$\log \|u(t)\| \leqq \log \|u(\tau_0)\| \frac{k(\tau_1) - k(t)}{k(\tau_1) - k(\tau_0)} + \log \|u(\tau_1)\| \frac{k(t) - k(\tau_0)}{k(\tau_1) - k(\tau_0)} + N,$$

where $k(\tau) = \int_{\tau_0}^{\tau} \exp(\mp \int_{\tau_0}^{s} M(\lambda) \, d\lambda) \, ds$, $M(\lambda)$ is a nonnegative, continuous, and integrable function on $[\tau_0, \tau_1)$ that is independent of u, the $+$ sign is used if $\|u(\tau_0)\| \geqq \|u(\tau_1)\|$ and the $-$ sign if $\|u(\tau_0)\| \leqq \|u(\tau_1)\|$, and N is a nonnegative constant independent of u.

PROOF.   This result follows from Agmon and Nirenberg's work [**1**, Théorème 2.1, Théorème 2.2, Proposition 2.1, Proposition 2.2] after verification of the three principal hypotheses of Agmon and Nirenberg in each of (a)–(d).

   A-N 0. $B = B_+ + B_-$, where $(B_+ v, w) = (v, B_+ w)$ and $\text{Re}(B_- v, v) = 0$ for all v, w in $C_0(\Omega)$ and each t in $[t_0, t_1)$, and each of B_+ and B_- are smooth;

   A-N I. $\text{Re}(B_+ u, B_- u) \geqq -C(t)(\|B_+ u\| + \|u\|)$; and

   A-N II. $(d/dt)(B_+ u, u) - 2\text{Re}(B_+ u, u_t) \geqq C(t)\|u\|(\|B_+ u\| + \|u\|)$, or alternately, $\|(B_+)_t u\| \leqq C(\|B_+ u\| + \|u\|)$.

   We may assume again that $A \equiv 0$ by a technique due to Egorov (see [**6**], [**7**]). Then set $B_+ = \frac{1}{2}(B + B^*)$ and $B_- = \frac{1}{2}(B - B^*)$, where $(B^* v, w) \equiv (v, Bw)$ for all smooth v and w and the hypotheses of Agmon-Nirenberg are easily seen to be satisfied.   Q.E.D.

REMARK 1.   Hypothesis (c) includes the case where the principal symbol b of B is independent of t.

REMARK 2. The convexity inequality is equivalent to the following formula:

$$\|u(t)\| \leqq e^N \|u(\tau_0)\|^{1-\alpha(t)} \|u(\tau_1)\|^{\alpha(t)},$$

where $0 \leqq \alpha(t) \nearrow 1$ for $\tau_0 \leqq t \nearrow \tau_1$, and $\alpha(t)$ is an integral expression. If $N=0$ and α is linear in t, then this would be a standard convexity result:

$$\|u(t)\| \leqq \|u(\tau_0)\|^{(\tau_1-t)/(\tau_1-\tau_0)} \|u(\tau_1)\|^{(t-\tau_0)/(\tau_1-\tau_0)},$$

or

$$\log \|u(t)\| \leqq ((\tau_1 - t)/(\tau_1 - \tau_0))\log \|u(\tau_0)\|$$
$$+ ((t - \tau_0)/(\tau_1 - \tau_0))\log \|u(\tau_1)\|.$$

We may relax the restrictions on B in Theorem 1 slightly.

PROPOSITION 2. *Suppose that, instead of satisfying one of* (a)–(d), *B fulfills*

(e) $B(x, t, D_x) = h(t)B_0(x, D_x)$, *where $h(t)$ is a continuous nonnegative function of t alone with zeros of finite order in $[t_0, t_1)$ such that $\lim_{t \nearrow t_1} h(t) \neq 0$, $B_0(x, D_x)$ is an arbitrary scalar Hermitian first order pseudo-differential operator and, in addition to $\|f\| \leqq \Phi(t)\|u\|$, the function f satisfies $f = O(h(t))$ for $h(t) \searrow 0$ and f/h is integrable with respect to t for each fixed x. Then the results of Theorem 1 remain valid.*

PROOF.   We suppose without loss of generality that $h(t_0)=0$, as we can always subdivide the t-interval and prove the result for each subinterval if $h(t)=0$ for some $t \in (t_0, t_1)$ and $h(t_0) \neq 0$. [If $h(t)$ is bounded away from 0 in $[t_0, t_1)$, then there is nothing to show.] We have

(2) $u_t(x, t) - h(t)B_0(x, D_x)u(x, t) = f(x, t).$

Let $s(t) = \int_{\tau_0}^t h(\lambda) \, d\lambda$. Then s is a differentiable monotone increasing function on $[t_0, t_1)$ and thus has a continuous monotone increasing inverse $t = \phi(s)$. We also define $v(x, s) = u(x, t)$. Then $v(x, s)$ is C^∞ in $\Omega \times [0, s(\tau_1))$ and v vanishes outside Ω for each fixed s. Since $u(x, t) = v(x, \int_{\tau_0}^t h(\lambda) \, d\lambda)$, we get

$$u_t(x, t) = v_s\left(x, \int_{\tau_0}^t h(\lambda) \, d\lambda\right)h(t)$$

by the chain rule.

Then (2) is equivalent to

$$v_s(x, s) - B_0(x, D_x)v(x, s) = f(x, t)/h(t) \equiv f_1(x, s).$$

Now, $f(x, t) = O(h(t))$ for $h(t) \searrow 0$ implies that $\|f_1(x, s)\| \leqq \Phi_1(s)\|u(s)\|$, where Φ_1 satisfies the same properties as Φ, and $B_0(x, D_x)$ satisfies (c) in Theorem 1: the principal symbol g of B_0 is nonnegative. Therefore the results of the theorem hold with s replaced by t, and, since s is merely a rescaling of the t-axis, we obtain the conclusions we desire.   Q.E.D.

Two types of "pseudo-parabolic" equations can also be handled by the technique of Agmon-Nirenberg.

THEOREM 3. *Suppose that instead of one of* (a)–(d) *holding, B is a Hermitian scalar-valued elliptic second order pseudo-differential operator. Then the conclusions of Theorem 1 are again valid.*

PROOF. This time we satisfy an alternative set of conditions. A-N 0, A-N I′, and A-N II, where

AN I′. $\|B_-u\|^2 \leq C(t)\|u\|(\|B_+u\| + \|u\|)$ replaces A-N I. We define B_+ and B_- as before: $B_+ = \frac{1}{2}(B+B^*) = B+H$, where H is a first order pseudo-differential operator, and $B_- = \frac{1}{2}(B-B^*) = -H$. Then A-N 0 is satisfied by definition. Since B is elliptic of order two, we have

$$\|\Lambda^2 u\| \leq C_0\|Bu\| + C\|\Lambda u\|, \qquad \text{where } \Lambda = (1-\Delta)^{1/2},$$
$$\leq C_0\|B_+u\| + C\|Hu\| + C\|\Lambda u\|$$
$$\leq C_0\|B_+u\| + C_1\|\Lambda u\|.$$

Given $\varepsilon > 0$, we can choose $C_\varepsilon = 1/\varepsilon$ such that $\|\Lambda u\| \leq \varepsilon\|\Lambda^2 u\| + C_\varepsilon\|u\|$. Choose ε so that $C_1\varepsilon < \frac{1}{2}$ for t in $[t_0, t_1)$. Then

$$\|\Lambda^2 u\| \leq C_0\|B_+u\| + C_1\varepsilon\|\Lambda^2 u\| + C_\varepsilon\|u\| + C_2\|u\|;$$

i.e., $\|\Lambda^2 u\| \leq 2C_0\|B_+u\| + 2(C_\varepsilon + C_2)\|u\|$. Thus

$$\|B_-u\|^2 = (Hu, Hu) \leq C(\Lambda u, \Lambda u) = C|(\Lambda^2 u, u)|$$
$$\leq \|\Lambda^2 u\| \|u\| \leq C\|u\| (\|B_+u\| + \|u\|),$$

which is A-N I′.

For A-N II,

$$\|(B_+)_t u\| \leq C\|\Lambda^2 u\| \leq C\|B_+u\| + C\|u\|.$$

Therefore the hypotheses of Agmon-Nirenberg are satisfied and the theorem holds.   Q.E.D.

The next result differs from Theorem 3 in that we allow second order differentiation in t, but use only first order pseudo-differential operators.

THEOREM 4. *Suppose that*

$$\|u_t - B_1 u\| \leq \Phi(t)\{\|u_t\|^2 + \mathrm{Re}(B_1 u, u) + M_B(t)(u, u)\}^{1/2},$$

for $u \in C^\infty(\Omega \times [t_1, t_2))$ with compact support in Ω for each fixed t, where B_1 is a real first order positive semidefinite pseudo-differential operator, $\Phi(t)$ is as before, and the function $M_B(t)$, which is continuous and integrable in t and satisfies $M_B(t) > 1$, comes from Gårding's inequality [4]: $\mathrm{Re}(B_1 u, u) \geq -(M_B - 1)\|u\|^2$. *Set $B = \frac{1}{2}(B_1 + B_1^*) + M_B$ and assume moreover that $|(B_t u, u)| \leq C|(Bu, u)|$ and $\|B_t u\| \leq C\|Bu\|$.*

If $\|u_t\|^2 + (Bu, u) = 0$ *for some* $\bar{t} \in [t_1, t_2)$, *then* $u_t \equiv 0$ *and* $(Bu, u) \equiv 0$ *for all t in the interval. This implies that* $u(t) = constant$, *and since B is positive definite, then* $u \equiv 0$ *in* $\Omega \times [t_1, t_2)$.

PROOF.   This is a direct result of Théorème 3.1 and Corollaire 3.1 in [1].

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