PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 36, Number 1, November 1972

A GENERALIZATION OF PEANO'S EXISTENCE THEOREM AND FLOW INVARIANCE

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ABSTRACT. Let $F\subseteq R^n$ be closed and $A:F\to R^n$ be continuous. Assuming that for $y\in F$ the distance from $y+hAy$ to F is $o(h)$ as $h\downarrow 0$, it is shown that for each $x\in F$ the Cauchy problem $u'=Au$, $u(0)=x$, has a solution $u:[0, T_x]\to F$ on some interval $[0, T_x]$, $T_x>0$.

Throughout this note F is a subset of R^m, $A:D(A)\to R^m$ is a continuous function with domain $D(A)$, and $F\subseteq D(A)\subseteq R^m$. $B_r(x)$ is the closed ball of radius r and center x in R^m. We will always assume that F is locally closed, i.e. for each $x\in F$ there is an $r>0$ such that $F\cap B_r(x)$ is closed in R^m. The euclidean norm of $y\in R^m$ is denoted by $|y|$ and $|y, F|$ stands for the distance from y to F. Our main result concerns the Cauchy problem

$$u' = Au, \qquad u(0) = x. \tag{1}$$

By a solution of (1) on an interval $[0, a]$, $a>0$, we mean a continuously differentiable function $u:[0,a]\to D(A)$ such that $u(0)=x$ and $u'(t)=Au(t)$ for $0\leqq t\leqq a$.

THEOREM 1. *Let*

$$\lim_{h\downarrow 0} h^{-1}|z + hAz, F| = 0 \quad \text{for } z\in F. \tag{A_1}$$

Then for each $x\in F$ there is a positive number T and a solution u of (1) *on* $[0, T]$ *such that $u(t)\in F$ for $0\leqq t\leqq T$.*

This theorem is related to results of Bony [1] and Brezis [2]. We say that F is forward invariant for (1) if whenever u is a solution of (1) on $[0, a]$, $a>0$, and $x\in F$, then $u(t)\in F$ for $0\leqq t\leqq a$. If $D(A)$ contains a neighborhood of F and F is forward invariant for (1), Theorem 1 is an obvious consequence of the Peano existence theorem. Brezis established that F is forward invariant for (1) if F is closed, (A_1) holds and A is Lipschitz continuous. Bony replaced (A_1) with a subtler condition. Let $x\in F$ and $y\in R^m$. If the interior of $B_{|y-x|}(y)$ does not meet F we say that $y-x$ is a normal to F at x in the sense of Bony. Let $\nu(x)$ be the set of such

Received by the editors April 25, 1972.

AMS 1970 *subject classifications*. Primary 34G05.

[1] Sponsored by the U.S. Army under Contract No. DA-31-124-ARO-D-462.

normals for $x \in F$. Bony showed that if F is closed, A is Lipschitz continuous and

$$(\mathrm{A}_2) \qquad (z, Ax) \leqq 0 \quad \text{for } x \in F \text{ and } z \in \nu(x),$$

then F is forward invariant for (1). Here (,) denotes the euclidean inner-product. Theorem 1 goes beyond such considerations in that it asserts the existence of solutions u with values in F in a generality which allows F to fail to be forward invariant for (1). Moreover, if F is closed and solutions of (1) for $x \in F$ are locally (forward) unique, then it follows at once from Theorem 1 that F is forward invariant for (1). This result generalizes the results of Brezis and Bony. Moreover, it settles a problem mentioned in Redheffer [**5**]. The results of Brezis and Bony are extended and simplified in [**5**].

Our proof is a simple adaptation of the method of polygonal approximation. The success of the argument rests on Lemma 1 which shows that if (A_1) holds, then it holds uniformly on compact subsets of F. It is obvious that $(\mathrm{A}_1) \Rightarrow (\mathrm{A}_2)$. Moreover, it is clear that if the conclusion of Theorem 1 holds, then (A_1) holds (see, e.g., [**2**]). Lemma 1 actually asserts that if (A_2) holds, then (A_1) holds uniformly on compact subsets of F, which does not seem obvious. As regards Theorem 1, see §5 of [**4**].

In [**4**] R. H. Martin carries out related investigations in a more complex infinite dimensional setting under assumptions which guarantee uniqueness of solutions of (1).

Proof of Theorem 1. Let $x \in F$ and $F_r = B_r(x) \cap F$. Choose $r > 0$ so that F_{2r} is closed and set

$$M = \max(\max\{|Ax| : x \in F_{2r}\}, 1), \qquad T = r/3M. \tag{2}$$

For each integer $n > 0$, set $x_{n,0} = x$ and inductively choose $x_{n,i} \in F$, $1 \leqq i \leqq n$, satisfying

$$2\,|x_{n,i} + (T/n)Ax_{n,i}, F| \geqq |x_{n,i+1} - (x_{n,i} + (T/n)Ax_{n,i})|. \tag{3}$$

The existence of $\{x_{n,i}\}_{i=0}^{n}$ is obvious once we show that if $x_{n,i}$ is defined for $0 \leqq i \leqq k \leqq n$ and $x_{n,i} \in F_r$ for $0 \leqq i < k$, then $x_{n,k} \in F_r$. If $z \in F_{2r}$, then

$$|z + (T/n)Az, F| \leqq |(z + (T/n)Az) - z| \leqq (T/n)M.$$

It then follows from (3) that

$$|x_{n,i+1} - x_{n,i}| \leqq (3T/n)M \tag{4}$$

for $0 \leqq i \leqq k-1$. Hence

$$|x_{n,k} - x| \leqq \sum_{i=0}^{k-1} |x_{n,i+1} - x_{n,i}| \leqq (k3T/n)M \leqq 3MT \leqq r.$$

To continue, define $u_n: [0, T] \to R^n$ by

$$u_n(t) = x_{n,i} + (t - iT/n)(n/T)(x_{n,i+1} - x_{n,i}) \tag{5}$$

for $iT/n \leqq t \leqq (i+1)T/n$, $0 \leqq i \leqq n-1$. Each u_n is a continuous piecewise linear mapping, $u_n(0)=x$, and by (4),

$$|u_n'(t)| \leqq 3M \tag{6}$$

for $0 \leqq t \leqq T$ and $t \notin \{iT/n : i=0, 1, \cdots, n\}$. It follows at once from the Arzela-Ascoli theorem that $\{u_n\}_{n=1}^{\infty}$ has a subsequence $\{u_{n(k)}\}_{k=1}^{\infty}$ which converges uniformly on $[0, T]$ to a limit $u(t)$. Clearly $|u_n(t), F| \leqq 3MT/n$ for $t \in [0, T]$, so $|u(t), F| \leqq 0$. Since $|u_n(t)-x| \leqq 3MT \leqq r$ for $0 \leqq t \leqq T$, $u(t) \in B_r(x)$. Thus $|u(t), F| = |u(t), F_{2r}|$. Since F_{2r} is closed $u(t) \in F$. Let D_r denote the right derivative. Since $D_r u_n(t) = (n/T)(x_{n,i+1} - x_{n,i})$ for $iT/n \leqq t < (i+1)T/n$ it will follow easily that $\lim_{k\to\infty} |D_r u_{n(k)}(t) - Au(t)| = 0$ holds uniformly in t, $0 \leqq t < T$, if we show

$$\lim_{n\to\infty} \max_{0 \leqq i \leqq n-1} |(n/T)(x_{n,i+1} - x_{n,i}) - Ax_{n,i}| = 0. \tag{7}$$

Thus the proof that $u(t)$ is a solution of (1) is completed by verifying (7), which is established with the aid of:

LEMMA 1. *Let A satisfy* (A_2). *Then* (A_1) *holds uniformly on every compact subset of F.*

PROOF. Let $C \subseteq F$ be compact. Since F is locally closed, C has a compact neighborhood K in R^n such that $K \cap F = C_1$ is compact. There is an $h_0 > 0$ such that $|x+hAx, F| = |x+hAx, C_1|$ for $x \in C$ and $0 \leqq h \leqq h_0$. We assume $0 \leqq s$, $\tau \leqq h_0$ everywhere below. Let $x \in C$ and $y_\tau = x + \tau Ax$. Choose $x_\tau \in C_1$ such that $|y_\tau, F| = |y_\tau - x_\tau|$. Set

$$\gamma(r) = \sup\{|Az - Ay| : z, y \in C_1 \text{ and } |z - y| \leqq r\} \tag{8}$$

and

$$f(\tau) = |y_\tau, F|^2. \tag{9}$$

Let $0 \leqq s < \tau \leqq h_0$. Then, using (8) and (9),

$$\begin{aligned} f(\tau) - f(s) &= |y_\tau - x_\tau|^2 - |y_s - x_s|^2 \leqq |y_\tau - x_s|^2 - |y_s - x_s|^2 \\ &= |y_\tau - y_s|^2 + 2(y_\tau - y_s, y_s - x_s) \\ &= (\tau - s)^2 |Ax|^2 + 2(\tau - s)(Ax_s, y_s - x_s) \\ &\quad + 2(\tau - s)(Ax - Ax_s, y_s - x_s) \\ &\leqq (\tau - s)^2 |Ax|^2 + 2(\tau - s)(Ax_s, y_s - x_s) \\ &\quad + 2(\tau - s)\gamma(|x - x_s|)\sqrt{(f(s))}. \end{aligned} \tag{10}$$

Next observe that $y_s - x_s \in \nu(x_s)$ and

$$|x - x_s| \leqq |x - y_s| + |y_s - x_s| \leqq 2\,|x - y_s| = 2s\,|Ax|.$$

Hence (by (A_2)) $(Ax_s, y_s - x_s) \leqq 0$ and $\gamma(|x - x_s|) \leqq \gamma(2s|Ax|)$. Using these estimates in (10), dividing by $(\tau - s)$ and letting $s \uparrow \tau$ yields

$$\limsup_{s \uparrow \tau} \frac{f(\tau) - f(s)}{\tau - s} \leqq 2\gamma(2\tau\,|Ax|)\sqrt{(f(\tau))}.$$

It follows at once that

$$\sqrt{(f(h))} = |x + hAx, F| \leqq \int_0^h \gamma(Ls)\,ds \tag{11}$$

where $L = 2\max\{|Ax| : x \in C_1\}$. Since A is uniformly continuous on C_1, $\lim_{r\downarrow 0} \gamma(r) = 0$. Thus the right-hand side of (11) is $o(h)$ as $h \downarrow 0$, and the proof is complete.

We finish the proof of Theorem 1. Set $\eta(\tau) = \sup\{|y + \tau Ay, F| : y \in F_r\}$. Since $(A_1) \Rightarrow (A_2)$, Lemma 1 implies that $\lim_{\tau\downarrow 0} \eta(\tau)/\tau = 0$. Since $x_{n,i} \in F_r$ for $0 \leqq i \leqq n$, (3) implies

$$2\eta(T/n)n/T \geqq |(n/T)(x_{n,i+1} - x_{n,i}) - Ax_{n,i}|$$

and (7) follows at once. The proof is complete.

REMARK 1. Let $E \subseteq R^m$ and $B : [a, b) \times E \to R^m$ be continuous and satisfy

$$\lim_{h\downarrow 0} h^{-1}\,|y + hB(t, y), E| = 0 \quad \text{for } t \in [a, b),\ y \in E. \tag{B_1}$$

Let $F = [a, b) \times E$ and $A : F \to R^{m+1}$ be defined by $A(t, y) = (1, B(t, y))$ for $(t, y) \in F$. If $t \in [a, b)$,

$$|(t, y) + hA(t, y), F| = |y + hB(t, y), E|$$

provided $h \geqq 0$ is sufficiently small. Thus A satisfies the assumptions of Theorem 1. Moreover, if E is locally closed (or closed), F is locally closed. In the usual way, we conclude:

THEOREM 2. *Let E be locally closed and $B : [a, b) \times E \to R^m$ be continuous and satisfy* (B_1). *If $(\tau, z) \in [a, b) \times E$, then there is a T, $0 < T < b - \tau$, and a continuously differentiable function $u : [\tau, \tau + T] \to E$ such that $u(\tau) = z$ and $u'(t) = B(t, u(t))$ for $t \in [\tau, \tau + T]$.*

REMARK 2. The proof of Lemma 1 shows that $(A_2) \Rightarrow (A_1)$ if F is a locally weakly closed subset of a Hilbert space H and $A : F \to H$ is continuous.

REMARK 3. The author proved Lemma 1 in 1971 in response to a query of R. Redheffer concerning the implication $(A_2)\Rightarrow(A_1)$. The proofs of Lemma 1 of the current paper and Theorem 1 of [5], which are related, were obtained independently.

REMARK 4. After the preparation of this note, P. Hartman kindly provided the author with a preprint of the paper [3] which contains, among other interesting results, a version of Theorem 1. Hartman's proof of Theorem 1 is shorter than that given here, but it does not include the fact that Theorem 1 is valid under the assumption (A_2) in place of (A_1). Certain convexity assumptions employed in [3] can be eliminated by using the normals $\nu(x)$ employed here and Lemma 1. For this purpose, it is worth noting that $\nu(x)$ is closed, convex and $r\nu(x)\subset\nu(x)$ for $0\leqq r\leqq 1$.

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