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THE ISOMETRIES OF $H^{\infty}(K)$

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ABSTRACT. Let K be a finite-dimensional Hilbert space. In this article a characterization is given of the linear isometries of the Banach space $H^{\infty}(K)$ onto itself. It is shown that T is such an isometry iff T is of the form $(TF)(z)=\mathscr{T}F(t(z))$, for $F \in H^{\infty}(K)$ and z belonging to the unit disc, where t is a conformal map of the disc onto itself and $\mathscr{T}$ is an isometry of K onto K.

0. **Introduction.** Throughout this paper the letter K represents a finite-dimensional complex Hilbert space. We denote by $\langle\cdot,\cdot\rangle$ the inner product in K, and fix some orthonormal basis $\{e_1,\cdots,e_N\}$ of K. Let $H^{\infty}(K)$ be the Banach space of functions F defined on the unit circle to K such that the scalar function $\langle F, e\rangle$ belongs to H^{∞} of the circle for each $e \in K$, and such that $\|F\|_{\infty}=\operatorname{ess\,sup}\|F(e^{ix})\|$ is finite. (Here $\|\cdot\|_{\infty}$ denotes the norm in $H^{\infty}(K)$, and $\|\cdot\|$ that in K.)

If $F \in H^{\infty}(K)$, we define the H^{∞} coordinate functions f_n by $f_n(e^{ix})=\langle F(e^{ix}), e_n\rangle$. Then almost everywhere we have $\sum_{n=1}^{N}|f_n(e^{ix})|^2<\infty$, and $F(e^{ix})=\sum_{n=1}^{N} f_n(e^{ix})e_n$. Moreover, each $F \in H^{\infty}(K)$ may be extended (via a power series) to an analytic function $F(z)$ on the unit disc $D=\{z:|z|<1\}$, having boundary values a.e. which determine F on the circle. This analytic function coincides with the function obtained by extending to D, in the usual way, the coordinate functions in the expression $F=\sum_n f_n e_n$. Thus, whenever it is convenient to do so, we may think of $H^{\infty}(K)$ as a space of bounded, vector-valued, analytic functions defined on D.

In recent years considerable work has been directed toward the determination of what properties of the Hardy classes H^p, $1\leqq p\leqq\infty$, can be generalized to the analogous spaces $H^p(K)$ of vector-valued functions. An excellent account of what had been done along these lines through the year 1964 can be found in the book by Helson [2]. Here we investigate the isometries of $H^{\infty}(K)$, which have been described for H^{∞} (i.e. for one-dimensional K), by deLeeuw, Rudin, and Wermer [5], and quite independently by Nagasawa [6]. Although our results generalize those of [5] and [6], the proofs, of necessity, require a quite different approach, since

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the authors of [5] and [6] rely heavily on the fact that $H^\infty(K)$ is a Banach algebra when K is one-dimensional.

1. **Extreme points in $H^\infty(K)^*$.** Let Y denote the maximal ideal space of H^∞ of the circle, and U denote the closed unit ball in K. Let X be the compact Hausdorff space $Y\times U$. If $F=\sum_n f_n e_n \in H^\infty(K)$, consider the function $\hat{F}$: $Y\to K$ determined by $\hat{F}(y)=\sum_n \hat{f}_n(y)e_n$, where $f\to\hat{f}$ is the Gelfand representation of H^∞. (It is easy to see, by virtue of the density of D in Y, that the function $\hat{F}$ so defined is independent of our choice of orthonormal basis for K.) Next define the scalar function $\tilde{F}$ on X by $\tilde{F}(y, e)=\langle \hat{F}(y), e\rangle$, $(y, e)\in X$. Since each $\hat{f}_n$ is continuous on Y, it is clear that $\hat{F}$ is continuous from Y to K, and hence that $\tilde{F}$ is continuous on X. Thus if we let $M=\{\tilde{F}: F\in H^\infty(K)\}$, the following lemma is then evident.

LEMMA 1.1. *M is a closed subspace of $C(X)$, the space of all continuous, complex-valued functions on X, and the mapping $F\to\tilde{F}$ is a linear isometry of $H^\infty(K)$ onto M.*

We let B denote the Choquet boundary for H^∞, considered as a function algebra on its maximal ideal space Y [7].

LEMMA 1.2. *A linear functional $\tilde{F}^*\in M^*$ is an extreme point of the unit ball of M^* iff $\tilde{F}^*$ is of the form $\tilde{F}^*(\tilde{F})=\tilde{F}(y, e)$, for some $y\in B$ and some $e\in K$ with $\|e\|=1$.*

PROOF. Suppose that $\tilde{F}^*$ is extreme. Then it is well known that $\tilde{F}^*(\tilde{F})=\tilde{F}(y, e)$ for some $(y, e)\in X$ [**1**, p. 441]. (Note that for scalars λ with $|\lambda|\leqq 1$, $\lambda\tilde{F}(y, e)=\tilde{F}(y, \bar{\lambda}e)$.) We claim that $y\in B$. For if not, there exist elements $f_i^*\in(H^\infty)^*$, $i=1, 2$, with $\|f_i^*\|\leqq 1$, each distinct from the point evaluation at y, such that $\hat{f}(y)=\frac{1}{2}[f_1^*(f)+f_2^*(f)]$ for each $f\in H^\infty$. Define elements $\tilde{F}_i^*\in M^*$ by $\tilde{F}_i^*(\tilde{F})=f_i^*(\langle F, e\rangle)$, for $i=1,2$ and $\tilde{F}\in M$. Then it is easily seen that the $\tilde{F}_i^*$ are distinct elements of the unit ball of M^*, and that $\tilde{F}^*=\frac{1}{2}[\tilde{F}_1^*+\tilde{F}_2^*]$, contradicting the assumption that $\tilde{F}^*$ is extreme. Thus $y\in B$, and an analogous argument shows that $\|e\|=1$.

Conversely, suppose that $\tilde{F}^*$ is of the form specified in the lemma. If $\tilde{F}^*$ is not extreme as claimed, there exist functionals $\tilde{F}_i^*$, $i=1, 2$, both distinct from $\tilde{F}^*$ and belonging to the unit ball of M^*, such that for all $\tilde{F}\in M$, $\tilde{F}(y, e)=\frac{1}{2}[\tilde{F}_1^*(\tilde{F})+\tilde{F}_2^*(\tilde{F})]$. Thus by the Hahn-Banach theorem and the Riesz representation theorem, we can find norm-preserving extensions of the $\tilde{F}_i^*$ to regular complex Borel measures μ_i on X. Let μ_1 and μ_2 be any two such extensions. Then it is clear that neither μ_i can be a scalar multiple of the point mass at (y, e). Thus by the regularity of the μ_i, there exist a positive number ε, and an open set O in X, where $O=V\times W$, V an open neighborhood of y in Y and W an open neighborhood of e in U, such that $|\mu_i|\,(O)<1-\varepsilon$, for $i=1, 2$.

Next note that $e' \to |\langle e, e'\rangle|$ is a continuous function on $U-W$. Since $U-W$ is compact, this function attains a maximum equal to $1-\delta$, for some $\delta>0$, on this set.

Now let f be a function in H^∞ such that $1=\|f\|=\hat{f}(y)$, and $|\hat{f}|<\delta\varepsilon/2$ on $Y-V$. (Such an f exists since $y\in B$.) Then defining $F\in H^\infty(K)$ by $F=fe$, we would have

$$\begin{aligned}
1 &= \langle \hat{F}(y), e\rangle = \tilde{F}(y, e) = \frac{1}{2}\left[\int \tilde{F}\, d\mu_1 + \int \tilde{F}\, d\mu_2\right] \\
&= \frac{1}{2}\left[\int_O \tilde{F}\, d(\mu_1+\mu_2) + \int_{V\times(U-W)} \tilde{F}\, d(\mu_1+\mu_2) + \int_{(Y-V)\times U} \tilde{F}\, d(\mu_1+\mu_2)\right] \\
&\leqq \tfrac{1}{2}[(|\mu_1|+|\mu_2|)(O) + (1-\delta)(2-(|\mu_1|+|\mu_2|)(O)) + \delta\varepsilon] \\
&< 1-\delta\varepsilon/2 < 1.
\end{aligned}$$

This contradiction then completes the proof of the lemma.

If $y\in Y$ and e is any element of K, we define the element $L_{(y,e)}$ of $H^\infty(K)^*$ by $L_{(y,e)}(F)=\langle \hat{F}(y), e\rangle$, for $F\in H^\infty(K)$. Let S denote the set of extreme points of the unit ball of $H^\infty(K)^*$. The two previous lemmas then give:

THEOREM 1. *The set S consists of all functionals of the form $L_{(y,e)}$, where $y\in B$ and e is an element of K with $\|e\|=1$.*

2. **The isometries.** Throughout this section, T will denote a fixed isometry of $H^\infty(K)$ onto itself. For any element $e\in K$, we denote by $\boldsymbol{e}$ that element of $H^\infty(K)$ which is constantly equal to e.

LEMMA 2.1. *Let e be any nonzero vector in K, and define a map $\tau: B\to B$ by $y'=\tau(y)$ if*

$$T^* L_{(y,e)} = L_{(y',e')} \tag{1}$$

for some $e'\in K$. Then τ is a one-one map of B onto itself, and is independent of the choice of e in $K-\{0\}$. Moreover, for fixed $y\in B$, the set of all e' given by (1) *as e varies in $K-\{0\}$ is all of $K-\{0\}$.*

PROOF. If $n>1$, then since T^* preserves the set S, we have $T^*L_{(y,e_1)}=L_{(y_1,e_1')}$ and $T^*L_{(y,e_n)}=L_{(y_n,e_n')}$, for certain $y_1, y_n\in B$ and $e_1', e_n'\in K$ with $\|e_1'\|=\|e_n'\|=1$. Now $L_{(y,(e_1+e_n)/\sqrt{2})}\in S$, and we have

$$\begin{aligned}
T^*L_{(y,(e_1+e_n)/\sqrt{2})} &= T^*[(\sqrt{2})^{-1}L_{(y,e_1)} + (\sqrt{2})^{-1}L_{(y,e_n)}] \\
&= (\sqrt{2})^{-1}L_{(y_1,e_1')} + (\sqrt{2})^{-1}L_{(y_n,e_n')}.
\end{aligned}$$

If $y_1\neq y_n$, then it is easy to see that the norm of the right-hand side is $\sqrt{2}$, while that of the left-hand side is 1. Thus we must have $y_1=y_n$, which proves that τ is independent of the choice of e. The remaining assertions of the lemma now follow easily by considering the function T^{*-1}.

LEMMA 2.2. *For each $y \in B$, and each n with $1 \leqq n \leqq N$,*

$$T^* L_{(y,(Te_n)^\wedge(y))} = L_{(\tau(y),e_n)},$$

and the set $\{(Te_n)^\wedge(y): n=1, \cdots, N\}$ is an orthonormal basis for K.

PROOF. By Lemma 2.1, for each n there exists $\varphi_n \in K$ with $\|\varphi_n\|=1$, such that $T^* L_{(y,\varphi_n)} = L_{(\tau(y),e_n)}$. We thus have

$$\begin{aligned} 1 &= \langle e_n, e_n \rangle = \langle e_n(\tau(y)), e_n \rangle = L_{(\tau(y),e_n)}(e_n) \\ &= T^{*-1} L_{(\tau(y),e_n)}(Te_n) = L_{(y,\varphi_n)}(Te_n) = \langle (Te_n)^\wedge(y), \varphi_n \rangle. \end{aligned}$$

And since $\|(Te_n)^\wedge(y)\| \leqq 1$, we must have $\varphi_n = (Te_n)^\wedge(y)$. Thus

$$T^* L_{(y,(Te_n)^\wedge(y))} = L_{(\tau(y),e_n)}.$$

Next suppose that e_k, $k \neq n$, is a second element of the given orthonormal basis. Then $T^* L_{(y,\varphi_k)} = L_{(\tau(y),e_k)}$, where $\varphi_k = (Te_k)^\wedge(y)$. We thus have

$$\begin{aligned} \langle \varphi_n, \varphi_k \rangle &= \langle (Te_n)^\wedge(y), \varphi_k \rangle = L_{(y,\varphi_k)}(Te_n) \\ &= T^* L_{(y,\varphi_k)}(e_n) = L_{(\tau(y),e_k)}(e_n) = \langle e_n, e_k \rangle = 0, \end{aligned}$$

and hence $\{(Te_n)^\wedge(y): n=1, \cdots, N\}$ is an orthonormal basis.

LEMMA 2.3. *If $F=\sum_n f_n e_n \in H^\infty(K)$, then for all $y \in B$,*
(a) $(TF)^\wedge(y) = \sum_n (\hat{f}_n \circ \tau)(y)(Te_n)^\wedge(y)$, *and*
(b) $(T^{-1}F)^\wedge(y) = \sum_n (\hat{f}_n \circ \tau^{-1})(y)(T^{-1}e_n)^\wedge(y)$.

PROOF. Since for each $y \in B$, $\{(Te_n)^\wedge(y)\}$ is a basis for K, we can write $(TF)^\wedge(y) = \sum_n h_n(y)(Te_n)^\wedge(y)$, where the h_n are scalar functions defined on B. Now fixing y and n, let $\varphi_n = (Te_n)^\wedge(y)$. Then (using Lemma 2.2) we have

$$\begin{aligned} h_n(y) &= \langle (TF)^\wedge(y), \varphi_n \rangle = L_{(y,\varphi_n)}(TF) \\ &= T^* L_{(y,\varphi_n)}(F) = L_{(\tau(y),e_n)}(F) \\ &= \langle \hat{F}(\tau(y)), e_n \rangle = (\hat{f}_n \circ \tau)(y), \end{aligned}$$

proving (a). (b) then follows by interchanging the roles of T and T^{-1}, τ and τ^{-1}.

Throughout the remainder of this article, we denote by A the algebra consisting of the restrictions to B of all $\hat{f}$, for $f \in H^\infty$.

LEMMA 2.4. *For each n, Te_n is a constant element of $H^\infty(K)$ (i.e. an element of K).*

PROOF. Suppose that $Te_n=\sum_k f_{nk}e_k$, and that $T^{-1}e_n=\sum_k g_{nk}e_k$. Then for $y\in B$,

$$e_n(y)=(TT^{-1}e_n)^{\wedge}(y)=\left(T\left(\sum_k g_{nk}e_k\right)\right)^{\wedge}(y)$$
$$=\sum_k(\hat{g}_{nk}\circ\tau)(y)(Te_k)^{\wedge}(y)=\sum_j\left(\sum_k(\hat{g}_{nk}\circ\tau)(y)\hat{f}_{kj}(y)\right)e_j.$$

Hence we have

$$\left\langle\sum_k(\hat{g}_{nk}\circ\tau)(y)e_k,\sum_k\hat{f}^{*}_{kj}(y)e_k\right\rangle=\delta_{nj},$$

(where f^* denotes the complex conjugate of f). And since

$$\left\{\sum_k(\hat{g}_{nk}\circ\tau)(y)e_k:n=1,\cdots,N\right\}=\{(T^{-1}e_n)^{\wedge}(\tau(y)):n=1,\cdots,N\}$$

is an orthonormal basis for K (by Lemma 2.2 with T^{-1} replacing T), we conclude that $\sum_k(\hat{g}_{nk}\circ\tau)(y)e_k=\sum_k\hat{f}^{*}_{kn}(y)e_k$ for all n, and hence that $(\hat{g}_{nk}\circ\tau)(y)=\hat{f}^{*}_{kn}(y)$ for all n, k and all $y\in B$.

Next, for $y\in B$,

$$(T(g_{nk}e_k))^{\wedge}(y)=\sum_j(\hat{g}_{nk}\circ\tau)(y)\hat{f}_{kj}(y)e_j,$$

so that $|\hat{f}_{kn}(y)|^2=(\hat{g}_{nk}\circ\tau)(y)\hat{f}_{kn}(y)=\langle(T(g_{nk}e_k))^{\wedge}(y),e_n\rangle\in A$. That is $|\hat{f}_{kn}|^2|_B=\hat{f}|_B$, for some $f\in H^{\infty}$. And since $\hat{f}$ is real-valued on B, the fact that each complex homomorphism of H^{∞} has a positive representing measure on B [3, p. 181] then implies that f is a constant function, say $f(z)=\lambda$ for all $z\in D$. (Here we use the fact that B is the Šilov boundary for H^{∞}, as may be seen by a construction paralleling that found on p. 174 of [3], and by the characterization of the points of B given in Corollary 8.3 (2), p. 53, of [7].)

Finally we have, for $y\in B$,

$$(T(g^2_{nk}e_k))^{\wedge}(y)=\sum_j(\hat{g}_{nk}\circ\tau)^2(y)\hat{f}_{kj}(y)e_j,$$

so that $\langle(T(g^2_{nk}e_k))^{\wedge}(y),e_n\rangle=\lambda\hat{f}^{*}_{kn}(y)\in A$, and hence $\hat{f}^{*}_{kn}|_B\in A$. Again by consideration of representing measures, we conclude that f_{kn} is a constant function, all k and n.

THEOREM 2. *Every linear isometry of $H^{\infty}(K)$ onto itself is of the form $(TF)(z)=\mathscr{T}F(t(z))$, $F\in H^{\infty}(K)$, $|z|<1$, where $\mathscr{T}$ is an isometry of K onto K and t is a conformal map of the unit disc onto itself. Conversely, every map T of this form is an isometry of $H^{\infty}(K)$ onto itself.*

Proof. The converse is immediate. Thus suppose that T is an isometry of $H^\infty(K)$ onto itself. We define $\mathscr{T}$ on the basis vectors e_n by letting $\mathscr{T}e_n$ be the constant value of Te_n, and then extend $\mathscr{T}$ linearly to K.

Now define $\Phi: H^\infty \to A$ by $\Phi(f)=\hat{f}|_B$. Let $\varphi_1=\mathscr{T}e_1$ and define $\Psi: A\to A$ by $(\Psi\hat{f})(y)=\langle (T(fe_1))\hat{\ }(y), \varphi_1\rangle$ for $y\in B$. By Lemma 2.3(a), $(\Psi\hat{f})(y)=(\hat{f}\circ\tau)(y)$, and thus $\hat{f}\circ\tau=\Psi\hat{f}\in A$. Thus to show that Ψ maps A onto itself, it suffices to show that given $f\in H^\infty$, then $\hat{f}\circ\tau^{-1}\in A$. But by Lemma 2.3(b), $(T^{-1}(f\varphi_1))\hat{\ }(y)=(\hat{f}\circ\tau^{-1})(y)e_1$, so that

$$(\hat{f}\circ\tau^{-1})(y) = \langle (T^{-1}(f\varphi_1))\hat{\ }(y), e_1\rangle \in A.$$

Since Φ and Ψ are obviously multiplicative, $\Phi^{-1}\Psi\Phi$ is an algebra automorphism of H^∞ onto itself, and hence by a result of Kakutani [4], for $f\in H^\infty$, $\Phi^{-1}\Psi\Phi(f)=f\circ t$, where t is a conformal map of the disc onto itself. Once more employing Lemma 2.3, we find that $(TF)(z)=\mathscr{T}(F\circ t)(z)$, $F\in H^\infty(K)$ and $z\in D$, as claimed.

Remarks. Much of the vectorial H^p theory, as developed in [2], is valid for arbitrary separable Hilbert space K. Thus it is natural to ask if Theorem 2 is true when K is infinite dimensional. The proofs given in §2 depend on finite dimensionality only to the extent that they depend on the characterization of extreme points in $H^\infty(K)^*$ obtained in §1. (Given that the extreme points are the same, a simple argument then shows that the orthonormal set provided by Lemma 2.2 is complete when K is infinite dimensional.) Thus, if indeed the extreme points should have the same characterization for infinite-dimensional K, the theorem holds in this case as well.

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