

## THE DIAGONAL FIBRATION, $H$ -SPACES, AND DUALITY<sup>1</sup>

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**ABSTRACT.** For any pointed topological space  $X$ , there is the fibration  $\Omega X \rightarrow X^I \rightarrow X \times X$  whose projection sends the map  $f: I \rightarrow X$  to  $(f(0), f(1))$ . We show that if  $X$  has the based homotopy type of a CW complex, then the above fibration is equivalent to one induced from the path space fibration  $\Omega X \rightarrow PX \rightarrow X$  if and only if  $X$  admits an  $H$ -space multiplication. Dually, we obtain a characterization of simply-connected co- $H$ -spaces.

**1. Notations and preliminaries.** If  $X$  is a topological space with base point  $*$ ,  $X^I$  denotes the space of all continuous maps of the closed unit interval  $I$  into  $X$ , with the compact-open topology.  $PX$ , the space of based paths in  $X$ , is the subspace of  $X^I$  consisting of all  $f$  such that  $f(0) = *$ .  $\Omega X$ , the loop space of  $X$ , is the subspace of  $X^I$  consisting of all  $f$  such that  $f(0) = f(1) = *$ . We have the well-known path space fibration  $\Omega X \rightarrow PX \rightarrow X$  whose projection sends  $f$  to  $f(1)$ .

**DEFINITION 1.1.** A fibration  $\Omega X \rightarrow E \rightarrow B$  is *induced* if there exist maps  $f: \Omega X \rightarrow \Omega X$ ,  $g: E \rightarrow PX$ ,  $h: B \rightarrow X$  such that the diagram

$$\begin{array}{ccccc} \Omega X & \longrightarrow & E & \longrightarrow & B \\ f \downarrow & & g \downarrow & & h \downarrow \\ \Omega X & \longrightarrow & PX & \longrightarrow & X \end{array}$$

commutes, and  $f$  is a weak homotopy equivalence.

If  $E$ ,  $B$ ,  $X$  have the homotopy types of connected CW complexes, this definition is easily seen to be equivalent to that of [1, p. 445].

$CX$  and  $\Sigma X$  denote the reduced cone and suspension, respectively, on  $X$ . We have the cofibration  $X \rightarrow CX \rightarrow \Sigma X$ .

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DEFINITION 1.2. A cofibration  $A \rightarrow Y \rightarrow \Sigma X$  is *induced* if there exist maps  $f: X \rightarrow A$ ,  $g: CX \rightarrow Y$ ,  $h: \Sigma X \rightarrow \Sigma X$  such that the diagram

$$\begin{array}{ccccc} X & \longrightarrow & CX & \longrightarrow & \Sigma X \\ f \downarrow & & g \downarrow & & \downarrow h \\ A & \longrightarrow & Y & \longrightarrow & \Sigma X \end{array}$$

commutes, and  $h$  is a weak homotopy equivalence.

It is easily seen that a cofibration which is induced in the sense of [1, p. 453] is also induced in our sense. If  $A$ ,  $Y$ ,  $X$  have the homotopy types of connected CW complexes, and if  $A$  is simply-connected, the two definitions of induced are easily seen to be equivalent.

2. The diagonal fibration and  $H$ -spaces.

DEFINITION 2.1. Let  $X$  be a pointed space. The fibration

$$\mathcal{D}_X: \Omega X \rightarrow X^I \xrightarrow{p} X \times X$$

defined by  $p(f) = (f(0), f(1))$  is called the *diagonal fibration* for  $X$ .

$\mathcal{D}_X$  is the fibration obtained by converting the diagonal map  $d: X \rightarrow X \times X$  to a fibration. In fact the inclusion  $i: X \rightarrow X^I$ , given by  $i(x) = \text{constant map with value } x$ , is a homotopy equivalence and the diagram

$$(2.2) \quad \begin{array}{ccc} X & \xrightarrow{i} & X^I \\ d \searrow & & \swarrow p \\ & X \times X & \end{array}$$

commutes.

The following lemma is an immediate observation:

LEMMA 2.3. Let  $i_1, i_2: X \rightarrow X \times X$  be given by  $i_1(x) = (x, *)$ ,  $i_2(x) = (*, x)$ . Then the induced fibrations  $i_1^* \mathcal{D}_X, i_2^* \mathcal{D}_X$  over  $X$  are both strictly equivalent to the path space fibration  $\Omega X \rightarrow PX \rightarrow X$ .

THEOREM 2.4. Let  $X$  be a pathwise-connected pointed space having the based homotopy type of a CW complex. Then the diagonal fibration  $\mathcal{D}_X$  is induced in the sense of 1.1 if and only if  $X$  admits an  $H$ -space multiplication.

PROOF. Suppose  $X$  admits an  $H$ -space multiplication  $m: X \times X \rightarrow X$ . Since  $X$  has the homotopy type of a CW complex, it follows from [3, Corollary 1.3] that this  $H$ -space structure has a right inversion, i.e. there is a map  $\eta: X \rightarrow X$  such that the composition

$$X \xrightarrow{d} X \times X \xrightarrow{\text{id} \times \eta} X \times X \xrightarrow{m} X$$

is homotopically trivial. Define  $h: X \times X \rightarrow X$  to be the composition  $m(\text{id} \times \eta)$ . Since  $hd$  is homotopically trivial, it follows from 2.2 and the fact that  $i$  is a homotopy equivalence that  $hp$  is homotopically trivial. Hence  $h$  is covered by a fibre-preserving map  $g: X^I \rightarrow PX$ . This, together with 2.3, yields the commutative diagram

$$\begin{array}{ccccc}
 \Omega X & \longrightarrow & PX & \longrightarrow & X \\
 = \downarrow & & \downarrow & & i_1 \downarrow \\
 \Omega X & \longrightarrow & X^I & \longrightarrow & X \times X \\
 f \downarrow & & g \downarrow & & h \downarrow \\
 \Omega X & \longrightarrow & PX & \longrightarrow & X
 \end{array}$$

where  $f$  is obtained by restricting  $g$ . From the definition of an  $H$ -space multiplication, it follows that  $hi_1$  is homotopic to the identity map on  $X$ . Hence, from the map of homotopy sequences of the fibrations and the 5-lemma, it follows that  $f_*: \Pi_n(\Omega X) \rightarrow \Pi_n(\Omega X)$  is an isomorphism for all  $n$ , and so  $f$  is a weak homotopy equivalence. Hence  $\mathcal{D}_X$  is induced.

Conversely, if  $\mathcal{D}_X$  is induced, there would exist a commutative diagram

$$\begin{array}{ccccc}
 \Omega X & \longrightarrow & X^I & \longrightarrow & X \times X \\
 f \downarrow & & g \downarrow & & h \downarrow \\
 \Omega X & \longrightarrow & PX & \longrightarrow & X
 \end{array}$$

with  $f$  a weak homotopy equivalence. Composing with  $i_j, j=1, 2$ , we obtain by 2.3 commutative diagrams

$$\begin{array}{ccccc}
 \Omega X & \longrightarrow & PX & \longrightarrow & X \\
 = \downarrow & & \downarrow & & \downarrow hi_j \\
 \Omega X & \longrightarrow & PX & \longrightarrow & X
 \end{array}$$

It follows that  $hi_1$  and  $hi_2$  are weak homotopy equivalences. Since  $X$  is assumed to have the homotopy type of a CW complex,  $hi_1$  and  $hi_2$  are homotopy equivalences. Let  $g_j: X \rightarrow X$  be pointed homotopy inverses to  $hi_j, j=1, 2$ . Define  $m: X \times X \rightarrow X$  to be the composition

$$X \times X \xrightarrow{g_1 \times g_2} X \times X \xrightarrow{h} X.$$

It follows easily that  $mi_1$  and  $mi_2$  are both homotopic to the identity map on  $X$ , and so  $m$  is an  $H$ -space multiplication.

REMARK 2.5. I am grateful to the referee for pointing out that Theorem 2.4 is also a corollary of the work of Porter [5] and Sugawara [6]. The

techniques of these papers depend strongly on classifying spaces for  $H$ -spaces and the proofs do not dualize, while the present proof does dualize to yield Theorem 3.3 below.

**3. The codiagonal cofibration and co- $H$ -spaces.** I am grateful to Professor P. J. Hilton for pointing out that the argument given in 2.4 dualizes.

For a pointed space  $X$ ,  $X \vee X$  denotes the one-point union of two copies of  $X$ , and  $\varphi: X \vee X \rightarrow X$  denotes the folding, or codiagonal, map. Let  $M_\varphi$  denote the reduced mapping cylinder of  $\varphi$ , and regard  $X \vee X$  as a subspace of  $M_\varphi$  under the canonical inclusion. Note that  $M_\varphi / X \vee X$ , the space obtained by collapsing  $X \vee X$  to a point, is homeomorphic to the reduced suspension of  $X$ . Explicitly,  $M_\varphi$  is the quotient space obtained from  $(X \vee X) \times I / * \times I$  by identifying  $[(x, *), 1] \sim [(*, x), 1]$ , and

$$\Sigma X = X \times I / X \times 0 \cup X \times 1 \cup * \times I.$$

The map  $h: M_\varphi / X \vee X \rightarrow \Sigma X$  given by  $h[x, *, t] = [x, t/2]$ ,  $h[* , x, t] = [x, 1 - t/2]$  is a homeomorphism.

**DEFINITION 3.1.** The cofibration  $\mathcal{F}_X: X \vee X \rightarrow M_\varphi \rightarrow \Sigma X$  is called the *folding* or *codiagonal cofibration* for  $X$ .

Analogous to 2.3, we have the following lemma, whose proof is straightforward:

**LEMMA 3.2.** *Let  $p_1, p_2: X \vee X \rightarrow X$  denote the natural projections on the first and second factors, respectively. Then the induced cofibrations  $p_{1*}\mathcal{F}_X$  and  $p_{2*}\mathcal{F}_X$  under  $X$  are both strictly equivalent to the cofibration  $X \rightarrow CX \rightarrow \Sigma X$ .*

**THEOREM 3.3.** *Let  $X$  be a path-connected space having the based homotopy type of a CW complex. If  $X$  admits a co- $H$  comultiplication with a left or right coinverse, then the codiagonal cofibration  $\mathcal{F}_X$  for  $X$  is induced in the sense of 1.2. Conversely, if  $X$  is simply connected, and if  $\mathcal{F}_X$  is induced, then  $X$  admits a co- $H$  comultiplication.*

**PROOF.** The proof is a straightforward dualization of the proof of 2.4, with the following difference: since we are now dealing with cofibrations instead of fibrations, we obtain maps of homology sequences rather than homotopy sequences as in 2.4. The hypothesis of simple connectivity is needed to ensure that a map inducing an isomorphism of homology groups is a homotopy equivalence.

Finally, we remark that in view of [2, Proposition 3.6], if  $X$  is a simply connected co- $H$ -space, the hypothesis that there exist a right or left co-inverse is redundant.

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