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THE DIOPHANTINE APPROXIMATION OF CERTAIN CONTINUED FRACTIONS

CHARLES F. OSGOOD

ABSTRACT. Given a real number α defined by

$$\frac{1}{\varphi(1)+}\,\frac{1}{\varphi(2)+}\cdots,$$

where φ is a function from the natural numbers to the rational numbers larger than or equal to one which satisfies certain restrictions on the growth of the numerators and denominators of the numbers $\varphi(n)$, then a lower bound is found in terms of φ for the diophantine approximation of α.

We wish to consider the continued fraction

$$\frac{1}{\varphi(1)+}\,\frac{1}{\varphi(2)+}\cdots,$$

where $\varphi(n)$ and an auxiliary function $\psi(n)$ satisfy certain conditions. Suppose that $\varphi(n)$ and $\psi(n)$ are functions defined from the positive integers to, respectively, the positive rationals and the positive integers such that

(a) $$1 \leqq \varphi(j) \quad \text{for } j = 1, 2, \cdots;$$
$$\lim_{n\to\infty} \prod_{j=1}^{n} \varphi(j) = \infty, \quad \text{and}$$

(b) $$\limsup_{n\to\infty} (\log(2))(1 + n/2)\left(\log\left(\prod_{j=1}^{n-1} \varphi(j)\right)\right)^{-1} = \eta$$
for some $0 \leqq \eta < +\infty$;

(c) $$0 \leqq \limsup_{n\to\infty} (\log(\varphi(n+1)))\left(\log\left(\prod_{j=1}^{n} \varphi(j)\right)\right)^{-1} = \delta < +\infty;$$

(d) $$\text{all } \psi(n)\left(\prod_{j\in S} \varphi(j)\right) \in Z \text{ if } S \subset \{1, \cdots, n\}; \text{ and}$$

(e) $$\limsup_{n\to\infty} (\log(\psi(n)))\left(\log\left(\prod_{j=1}^{n} \varphi(j)\right)\right)^{-1} = \gamma < 1.$$

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(Since each $\varphi(j)>0$ and $\sum_{j=1}^{\infty}\varphi(j)\geqq\sum_{j=1}^{\infty}1=+\infty$, it follows immediately, from [**1**], that the continued fraction

$$\frac{1}{\varphi(1)+}\,\frac{1}{\varphi(2)+}\cdots\frac{1}{\varphi(n)+}\cdots$$

converges to a positive real number α.)

THEOREM. *For every $\varepsilon>0$, there exists a $c(\varepsilon)>0$ such that, for every pair of positive integers p and q,*

$$|\alpha-pq^{-1}|>c(\varepsilon)q^{-(1+\theta+\varepsilon)} \tag{1}$$

where $\theta=(1+\delta)(1+\eta+\gamma)(1-\gamma)^{-1}$.

EXAMPLES. We note that $\varphi(n)=n$ and $\psi(n)=1$ satisfy (a)–(e), as do $\varphi(n)=(ts^{-1})^n$ and $\psi(n)=s^{n(n+1)/2}$ for positive integers s and t with $t>s^2$. Also, with s and t as above, $\varphi(n)=p_d((ts^{-1})^n)$ and $\psi(n)=s^{n(n+1)d/2}$ satisfy (a)–(e) where $p_d(x)\in Z^+[x]$ and $p_d(x)$ has degree $d\geqq 1$ in x. One may generalize this to see that, where d_j, t_j, and s_j, for $1\leqq j\leqq m$, are positive integers satisfying

$$\prod_{j=1}^{m}t_j^{d_j}>\left(\prod_{j=1}^{m}s_j^{d_j}\right)^2,$$

where $\varphi(n)=p_{d_0,\cdots,d_m}(n,q_1^n,\cdots,q_m^n)$ is a polynomial in n, $q_1^n=(t_1s_1^{-1})^n,\cdots,$ $q_m^n=(t_ms_m^{-1})^n$ with positive integral coefficients and degrees $d_0,\cdots,d_m$, respectively, of the form a nonzero polynomial in n times $q_1^{nd_1}\cdots q_m^{nd_m}$ plus terms of lower degree in some q_j^n, and where $\psi(n)=\prod_{j=1}^{m}s_j^{n(n+1)d_j/2}$ then hypotheses (a)–(e) of the Theorem are satisfied also.

One may drop the assumption of positive coefficients in the $p_d(x)$ above. To see this, note first that the dominant term in each $\varphi(n)$ forces it to be of constant sign for large enough n. Then, for some positive integer N, the hypotheses of the Theorem are satisfied upon substituting

$$\beta=\frac{1}{|\varphi(N)|+}\,\frac{1}{|\varphi(N+1)|+}\cdots\quad\text{for }\alpha.$$

The $(N+j)$th convergent of the original continued fraction for α is given by $(Ac_j+B)(Cc_j+D)^{-1}$, for integers A, B, C, and D with $\begin{vmatrix}A&B\\C&D\end{vmatrix}\neq 0$, where c_j is the jth convergent of the continued fraction above giving β and AC^{-1} and BD^{-1} are, respectively, the Nth and $(N-1)$st convergents of α. Since β is irrational (by (1)) eventually each $(Ac_j+B)(Cc_j+D)^{-1}$ is defined and, as $j\to\infty$, these convergents approach $(A\beta+B)(C\beta+D)^{-1}=T(\beta)$, where $T(z)=(Az+B)(Cz+D)^{-1}$.

If $|\beta-T^{-1}(pq^{-1})|<|\beta-DC^{-1}|$ we have, by the law of the mean, that $|T(\beta)-pq^{-1}|=|T(\beta)-T(T^{-1}(pq^{-1}))|=|T'(\xi)|\,|\beta-T^{-1}(pq^{-1})|$ where ξ is some point lying between β and $T^{-1}(pq^{-1})$. By the continuity of $T(z)$ at the irrational point β, we see the continuity of $T^{-1}(z)$ at $T(\beta)$ and this latter property implies that, by requiring $|T(\beta)-pq^{-1}|$ to be sufficiently small, we can guarantee $|\beta-T^{-1}(pq^{-1})|<|\beta-DC^{-1}|$. Thus either

$$\text{(i)} \qquad |T(\beta) - pq^{-1}| \geqq K_1\,|\beta - T^{-1}(pq^{-1})|$$

for some constant $K_1>0$ independent of p and q or

$$\text{(ii)} \qquad |T(\beta) - pq^{-1}| > K_2 > 0$$

for some constant K_2 independent of p and q. One may conclude, from (1) with $\alpha=\beta$ and the alternatives (i) and (ii) above, that (1) holds with $\alpha=T(\beta)$, for a $0<c_1(\varepsilon)\leqq c(\varepsilon)$ replacing $c(\varepsilon)$ in (1); since, if $c_1(\varepsilon)<K_2$, case (ii) implies that (1) holds and case (i) says that $|T(\beta)-pq^{-1}|\geqq K_1|\beta-T^{-1}(pq^{-1})|$, which is larger than $c_1(\varepsilon)q^{-(1+\theta+\varepsilon)}$ for some $0<c_1(\varepsilon)\leqq \min\{c(\varepsilon), K_2\}$ independent of p and q.

In all of the above examples, δ was zero. To see a case in which $\delta>0$ set $\varphi(n)=(5/2)^{2^n}$ and $\psi(n)=\frac{1}{2}(2^{2^{n+1}})$. One could generalize along the lines above and build up more complicated examples from this last example.

The author was led to consider the present problem after obtaining in **[2]**, by different but related methods, a lower bound on the simultaneous diophantine approximation of the real number

$$1+\frac{zq^{-1}}{1+}\,\frac{zq^{-2}}{1+}\cdots\frac{zq^{-n}}{1+},$$

where q denotes an integer, z denotes a rational number, $|q|>1$, and $|z|>0$. (For $z=1$, the above number equals

$$\prod_{n=0}^{\infty}(1-q^{-(5n+2)})(1-q^{-(5n+1)})^{-1}(1-q^{-(5n+3)})(1-q^{-(5n+4)})^{-1},$$

as was shown by Ramanujan.) Since we may rewrite this real number as

$$1+\frac{z}{q+}\,\frac{z}{q+}\,\frac{z}{q^2+}\,\frac{z}{q^2+}\cdots\frac{z}{q^n+}\,\frac{z}{q^n+}\,\frac{z}{q^{n+1}+}\cdots,$$

it follows that the present theory applies at $z=1$. In each case, we obtain inequality (1) with $\theta=1$.

PROOF OF THE THEOREM. Since $\sum_{j=2}^{\infty}\varphi(j)=+\infty$ we see that the continued fraction with these partial quotients converges to a positive real number, α'. Now $\alpha=(\varphi(1)+\alpha')^{-1}$ so we have $\alpha<(\varphi(1))^{-1}$. Using induction set $a_0=1$, $a_1=\alpha<(\varphi(1))^{-1}$, $a_2=a_1((a_1a_0^{-1})^{-1}-\varphi(1))<(\varphi(1)\varphi(2))^{-1},\cdots,$ $a_n=a_{n-1}((a_{n-1}a_{n-2}^{-1})^{-1}-\varphi(n-1))<(\varphi(1)\cdots\varphi(n))^{-1}$. Note that no a_j above is zero since each a_j is the product of nonzero real numbers. Also for $n=2,3,\cdots,$

$$a_n = -\varphi(n-1)a_{n-1} + a_{n-2}. \tag{2}$$

Using (2) repeatedly, we may write each $\psi(n-1)a_n$ as a linear form $L_n=A_na_1+B_na_0$ for integers A_n and B_n satisfying easily calculable upper bounds on their absolute values. We find that

$$|B_n| \leqq \sum_{0\leqq k\leqq n/2} \binom{n-k}{k}\left(\prod_{j=1}^{n-1}\varphi(j)\right)\psi(n-1),$$

where $\binom{n-k}{k}<2^{n-k}$. Thus $B_n<2^{1+n/2}(\prod_{j=1}^{n-1}\varphi(j))\psi(n-1)$. Similarly,

$$\begin{aligned}|A_n| &\leqq \sum_{0\leqq k\leqq n/2}\binom{n-1-k}{k}\left(\prod_{j=1}^{n-1}\varphi(j)\right)\psi(n-1)\\ &< 2^{1+n/2}\left(\prod_{j=1}^{n-1}\varphi(j)\right)\psi(n-1).\end{aligned}$$

Using hypotheses (b) and (e), for every $\varepsilon>0$,

$$\max\{|A_n|,|B_n|\} < \left(\prod_{j=1}^{n-1}\varphi(j)\right)^{(1+\eta+\gamma+\varepsilon)}$$

if n is sufficiently large. Under these same conditions we have, using (c), that

$$\max\{|A_n|,|A_{n+1}|,|B_n|,|B_{n+1}|\} < \left(\prod_{j=1}^{n}\varphi(j)\right)^{(1+\eta+\gamma+\varepsilon)}. \tag{3}$$

Also, if n is sufficiently large,

$$\max\{|L_n|,|L_{n+1}|\} \leqq \left(\prod_{j=1}^{n}\varphi(j)\right)^{-(1-\gamma)+\varepsilon}. \tag{4}$$

From (2), we may write

$$\begin{pmatrix}a_n\\ a_{n+1}\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & -\varphi(n)\end{pmatrix}\begin{pmatrix}a_{n-1}\\ a_n\end{pmatrix}.$$

The entries of

$$\begin{pmatrix}0 & 1\\ 1 & -\varphi(n)\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & -\varphi(n-1)\end{pmatrix}\cdots\begin{pmatrix}0 & 1\\ 1 & -\varphi(1)\end{pmatrix}$$

must be proportional to those of

$$\begin{pmatrix} A_n & B_n \\ A_{n+1} & B_{n+1} \end{pmatrix},$$

therefore

$$\begin{vmatrix} A_n & B_n \\ A_{n+1} & B_{n+1} \end{vmatrix} \neq 0.$$

For each real $1 \leqq x < \infty$ set

$$f(x) = \left(\left(\prod_{j=1}^{[x]} \varphi(j) \right) (\varphi([x]+1))^{x-[x]} \right)^{\theta_1}$$

where $[x]$ denotes the greatest integer function and $\theta_1 = (1+\delta)(1+\eta+\gamma) = (1-\gamma)\theta$. Note that $f(x)$ is monotone increasing, and that $f(x)$ takes the nonnegative reals onto $[\varphi(1), +\infty)$, since $\prod_{j=1}^{\infty} \varphi(j) = \infty$. For $n = 1, \cdots$ set

$$\Delta_n = \begin{pmatrix} A_n & B_n \\ A_{n+1} & B_{n+1} \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1 \\ \alpha \end{pmatrix}.$$

Let $\|\text{matrix}\|$ denote the maximum of the absolute values of its entries. Then given $0 < \varepsilon < 1$, if n_1 is sufficiently large we have, for all $n \geqq n_1$,

(i) $\|\Delta_n\| \leqq (f(n))^{1+\varepsilon/5}$, from (3),

(ii) $\|\Delta_n V\| \leqq (f(n))^{-\theta_2^{-1}(1-\varepsilon/5)}$, from (4), where $\theta_2 = (1+\eta+\gamma)(1-\gamma)^{-1}$, and

(iii) $f(n) \leqq (f(n-1))^{(1+\delta)(1+\varepsilon/5)}$ from (c).

We shall next state a lemma from which we will be able to conclude the present Theorem. Then we shall conclude this paper with a proof of the Lemma.

Suppose that for some positive integer t we have a sequence Δ_m of t by t nonsingular matrices over the Gaussian integers and a t by 1 matrix $V \neq 0$ with complex entries. Let $f(n)$ be a monotone increasing function from $[1, +\infty)$ onto $[c, +\infty)$ for some $c \geqq 1$. Let $0 < \varepsilon < +\infty$, $0 < r < +\infty$, and $(1+\varepsilon/r)^2(1-\varepsilon/r)^{-1} < 1+\varepsilon$. Suppose that, for all nonnegative integers $n \geqq n_1 \geqq 1$,

(i) $\|\Delta_n\| \leqq (f(n))^{1+\varepsilon/r}$,

(ii) $\|\Delta_n V\| \leqq (f(n))^{-\Lambda(1-\varepsilon/r)}$ for some $\Lambda > 0$, and

(iii) $f(n) \leqq (f(n-1))^{(1+\delta)(1+\varepsilon/r)}$.

Let q denote a nonzero Gaussian integer and P denote a general t by 1 matrix of Gaussian integers with not all entries zero.

LEMMA. *If* $|q| > \frac{1}{2}(f(n_1))^{\Lambda(1-\varepsilon/r)}$ *then* $\|V - Pq^{-1}\| \geqq t^{-1}|2q|^{-(1+(1+\varepsilon)(1+\delta)/\Lambda)}$.

Note that if $0<\varepsilon<1$ then $(1+\varepsilon/5)^2(1-\varepsilon/5)^{-1}<(1+\varepsilon/5)^2(1+\varepsilon/4)<(1+\varepsilon/4)^3<1+61(64)^{-1}\varepsilon<1+\varepsilon$. One may then apply the Lemma to our present situation with $\Lambda=\theta_2^{-1}=(1-\gamma)(1+\eta+\gamma)^{-1}=(1+\delta)\theta^{-1}$. Thus we see that

$$\|V - Pq^{-1}\| \geqq \tfrac{1}{2}\,|2q|^{-(1+(1+\varepsilon)\theta)}, \tag{5}$$

where $V=\binom{1}{\alpha}$. Setting $P=\binom{q}{p}$, where p is an arbitrary nonnegative integer, and letting q be an arbitrary positive integer, we obtain

$$|\alpha - pq^{-1}| > q^{-(1+(1+\varepsilon)\theta)} \tag{6}$$

if q is sufficiently large. Since (6) would be impossible if α were rational, we see that there must exist a $c(\varepsilon)>0$ such that $|\alpha-pq^{-1}|>c(\varepsilon)q^{-(1+\theta+\varepsilon)}$ if $q\geqq 1$. This proves our Theorem, assuming the Lemma.

PROOF OF THE LEMMA. For each nonnegative integer n,

$$\|\Delta_n(V - Pq^{-1})\| \geqq \|\Delta_n Pq^{-1}\| - \|\Delta_n V\|.$$

Then $\|\Delta_n Pq^{-1}\|\geqq|q|^{-1}$, and

$$\|\Delta_n(V - Pq^{-1})\| \geqq |q|^{-1} - \|\Delta_n V\|, \tag{7}$$

since each entry in P and Δ_n is a Gaussian integer, $P\neq 0$, and each Δ_n is nonsingular. We now choose n to be the first integer such that $|2q|<(f(n))^{\Lambda(1-\varepsilon/r)}$. Since $(f(n_1))^{\Lambda(1-\varepsilon/r)}<|2q|<(f(n))^{\Lambda(1-\varepsilon/r)}$ and $f(n)$ is monotone increasing we have $n>n_1$.

Therefore, we may use (ii) with (7) to conclude that

$$\|\Delta_n(V - Pq^{-1})\| \geqq |q|^{-1} - (f(n))^{-\Lambda(1-\varepsilon/r)}.$$

Since $|2q|^{-1}>(f(n))^{-\Lambda(1-\varepsilon/r)}$ we have

$$\|\Delta_n(V - Pq^{-1})\| \geqq |2q|^{-1}. \tag{8}$$

From our choice of n and from hypothesis (iii), we see that

$$\begin{aligned}\exp((\log|2q|)\Lambda^{-1}(1 + \varepsilon/r)^2(1 - \varepsilon/r)^{-1}(1 + \delta))\\ \geqq (f(n-1))^{(1+\varepsilon/r)^2(1+\delta)} \geqq (f(n))^{(1+\varepsilon/r)}.\end{aligned}$$

Since $(1+\varepsilon/r)^2(1-\varepsilon/r)^{-1}<1+\varepsilon$, we see that

$$(f(n))^{1+\varepsilon/r} \leqq |2q|^{(1+\varepsilon)(1+\delta)\Lambda^{-1}}. \tag{9}$$

From (8), hypothesis (i), and (9), we conclude that

$$\begin{aligned}\|V - Pq^{-1}\| &\geqq t^{-1}\,|2q|^{-1}\,\|\Delta_n\|^{-1} \geqq t^{-1}\,|2q|^{-1}(f(n))^{-(1+\varepsilon/r)}\\ &\geqq t^{-1}\,|2q|^{-(1+(1+\varepsilon)(1+\delta)\Lambda^{-1})}.\end{aligned}$$

This proves the Lemma.

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Mathematics Research Center, Naval Research Laboratory, Washington, D.C. 20390