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ON THE MEAN MODULUS OF TRIGONOMETRIC POLYNOMIALS AND A CONJECTURE OF S. CHOWLA

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ABSTRACT. Let $\{m_k\}$ be a strictly increasing sequence of positive integers. S. Chowla (1965) conjectured that

$$\min_{0\le x<1}\sum_{k=1}^{n}\cos 2\pi m_k x<-cn^{1/2},$$

$c>0$ being an absolute constant. Let $K_1, K_2, \cdots, K_N$ be the distinct integers m_l-m_k, $1\leqq k<l\leqq n$; r_j, $1\leqq j\leqq N$, the number of pairs (k, l) with $1\leqq k<l\leqq n$ and $m_l-m_k=K_j$; and

$$r(n)=\max_{1\le j\le N} r_j.$$

Lower bounds for $\int_0^1|\sum_{k=1}^n c_k e^{2\pi i m_k x}|\,dx$, c_k arbitrary complex numbers and $\int_0^1|\sum_{k=1}^n \gamma_k\cos 2\pi(m_k x+\alpha_k)|\,dx$, $\gamma_k\geqq 0$, α_k real, are obtained in terms of n, $r(n)$ and the c_k and γ_k respectively and it has been deduced that in case $r(n)=\delta$, independent of n, then

$$\min_{0\le x<1}\sum_{k=1}^{n}\cos 2\pi m_k x<-\frac{1}{2^{5/2}}\frac{1}{(\delta+1)^{1/2}}n^{1/2}.$$

1. **Introduction.** Throughout the following $\{m_k\}$ stands for a strictly increasing sequence of positive integers. Chowla **[2]** conjectured that for any sequence $\{m_k\}$

$$\min_{0\le x<1}\sum_{k=1}^{n}\cos 2\pi m_k x<-cn^{1/2}, \tag{1.1}$$

$c>0$ being an absolute constant. Uchiyama **[4]** proved that given any n distinct positive integers $m_1, m_2, \cdots, m_n$ there exists a subset $m_{j_1}, m_{j_2}, \cdots, m_{j_r}$ of $m_1, m_2, \cdots, m_n$ such that

$$\min_{0\le x<1}\sum_{k=1}^{r}\cos 2\pi m_{j_k} x<-\left(\frac{1}{4}\right)\left(\frac{1}{6}\right)^{1/2}n^{1/2}. \tag{1.2}$$

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In [3] Kurtz and Shah proved that

$$\min_{0\leq x<1} \sum_{k=1}^{n} \cos 2\pi m_k x < -\left(\frac{1}{8}\right) n^{1/2} \tag{1.3}$$

for a special class of sequences which they call admissible sequences. A sequence $\{m_k\}$ is called admissible if $m_k - m_j + m_l - m_p \neq 0$ if $k \neq j$, $k \neq p$, and $j \neq l$ all hold (see [3]).

The purpose of this paper is to prove (1.1) for much more general classes of sequences $\{m_k\}$. Specifically, let $N = N(n)$ be the number of distinct positive integers $m_l - m_k$, $1 \leqq k < l \leqq n$, and let these distinct integers be denoted by $K_1, K_2, \cdots, K_N$. Also, let for $1 \leqq j \leqq N$, r_j be the number of pairs (k, l) such that $1 \leqq k < l \leqq n$ and $m_l - m_k = K_j$, and $r(n) = \max\{r_1, r_2, \cdots, r_N\}$. We shall prove (see Corollary 2.4) that, if $r(n)$ is independent of n, say $r(n) = \delta$, then (1.1) is true with $c = (\frac{1}{2})^{5/2}(1/(\delta+1)^{1/2})$. We shall denote the class of all sequences $\{m_k\}$ with $r(n) = \delta$ by B_δ.

Let, for arbitrary complex sequence $\{c_k\}$,

$$S_n(x) = \sum_{k=1}^{n} c_k e(m_k x), \qquad e(x) = \exp(2\pi i x), \tag{1.4}$$

$$R_n = \sum_{k=1}^{n} |c_k|^2, \qquad T_n = \sum_{k=1}^{n} |c_k|^4 \tag{1.5}$$

$$L_j = \sum_{m_l - m_k = K_j; 1 \leq k < l \leq n} \mathrm{Re}(c_k \bar{c}_l), \qquad M_j = \sum_{m_l - m_k = K_j; 1 \leqq k < l \leqq n} \mathrm{Im}(c_k \bar{c}_l), \tag{1.6}$$

$$L = \sum_{j=1}^{N} L_j^2, \qquad M = \sum_{j=1}^{N} M_j^2, \tag{1.7}$$

and finally,

$$T_n(x, \alpha) = T_n(x) = \sum_{k=1}^{n} \gamma_k \cos 2\pi(m_k x + \alpha_k) \tag{1.8}$$

where $\gamma_k \geqq 0$, and α_k real.

Lower bounds for the mean modulus $\int_0^1 |S_n(x)|\, dx$ and $\int_0^1 |T_n(x)|\, dx$ are obtained (Theorems 1 and 2) for arbitrary sequences $\{m_k\}$. We note that our results contain the results of [3] for admissible sequences (see the remark following Corollary 2.4). For results of different type on the min $T_n(x)$, we refer to Theorem 3 of [1] and Theorem 4 of [3].

2. THEOREM 1. *For any sequence* $\{m_k\}$

$$\int_0^1 |S_n(x)|\, dx \geqq \frac{R_n^{1/2}}{\left(1 + r(n) - \dfrac{1}{n}\right)^{1/2}}, \qquad n = 1, 2, 3, \cdots.$$

COROLLARY 1.1. *For any sequence* $\{m_k\} \in B_\delta$

$$\int_0^1 |S_n(x)|\, dx \geqq \frac{R_n^{1/2}}{\left(1+\delta-\frac{1}{n}\right)^{1/2}}, \qquad n = 1, 2, 3, \cdots.$$

For the real series, we have

THEOREM 2. *For any sequence* $\{m_k\}$,

$$\int_0^1 |T_n(x)|\, dx \geqq \frac{1}{2^{3/2}} \frac{\left\{\sum_{k=1}^n \gamma_k^2\right\}^{1/2}}{\left(1+r(n)-\frac{1}{n}\right)^{1/2}}, \qquad n = 1, 2, 3, \cdots.$$

COROLLARY 2.1. *For any sequence* $\{m_k\} \in B_\delta$

$$\int_0^1 |T_n(x)|\, dx \geqq \frac{1}{2^{3/2}} \frac{\left\{\sum_{k=1}^n \gamma_k^2\right\}^{1/2}}{\left(1+\delta-\frac{1}{n}\right)^{1/2}}, \qquad n = 1, 2, 3, \cdots.$$

COROLLARY 2.2. *For any sequence* $\{m_k\}$,

$$\min_{0 \leqq x < 1} T_n(x) \leqq -\frac{1}{2^{5/2}} \frac{\left\{\sum_{k=1}^n \gamma_k^2\right\}^{1/2}}{\left(1+r(n)-\frac{1}{n}\right)^{1/2}}, \qquad n = 1, 2, 3, \cdots.$$

COROLLARY 2.3. *For any sequence* $\{m_k\} \in B_\delta$

$$\min_{0 \leqq x < 1} T_n(x) \leqq -\frac{1}{2^{5/2}} \frac{\left\{\sum_{k=1}^n \gamma_k^2\right\}^{1/2}}{\left(1+\delta-\frac{1}{n}\right)^{1/2}}, \qquad n = 1, 2, 3, \cdots.$$

COROLLARY 2.4. *For any sequence* $\{m_k\} \in B_\delta$

$$\min_{0 \leqq x < 1} \sum_{k=1}^n \cos 2\pi m_k x < -\frac{n^{1/2}}{2^{5/2}(1+\delta)^{1/2}}, \qquad n = 1, 2, 3, \cdots.$$

REMARK. It is not hard to see that if $\{m_k\}$ is admissible, $\delta = 1$ and our Corollaries 1.1, 2.1, 2.3, and 2.4 reduce to Theorems 1, 2, 3 and Corollary of [3].

3. We need only to prove Theorems 1 and 2 and the Corollary 2.2. The proofs of Theorem 2 and Corollary 2.2 are similar to the proofs of Theorems 2 and 3 respectively of [3]. We skip some of the details in the proof of Theorem 2 and the proof of Corollary 2.2, referring the reader to [3].

LEMMA 1. $\int_0^1 |S_n(x)|^4\,dx = R_n^2 + 2(L+M)$.

PROOF. We have

$$\begin{aligned}|S_n(x)|^2 &= \left\{\sum_{k=1}^{n} c_k e(m_k x)\right\}\left\{\sum_{k=1}^{n} \bar{c}_k e(-m_k x)\right\}\\ &= \sum_{k=1}^{n} |c_k|^2 + \sum_{k\neq l;1\leqq k,l\leqq n} c_k\bar{c}_l e((m_k-m_l)x)\\ &= R_n + \sum_{1\leqq k<l\leqq n}\{c_k\bar{c}_l e((m_k-m_l)x) + c_l\bar{c}_k e((m_l-m_k)x)\}\\ &= R_n + 2\sum_{1\leqq k<l\leqq n}\{\mathrm{Re}(c_k\bar{c}_l)\cos 2\pi(m_l-m_k)x\\ &\qquad + \mathrm{Im}(c_k\bar{c}_l)\sin 2\pi(m_l-m_k)x\}\end{aligned}$$

which, by (1.6), is

$$= R_n + 2\sum_{j=1}^{N}\{L_j\cos 2\pi K_j x + M_j \sin 2\pi K_j x\}. \tag{3.1}$$

Now, since the K_j, $1\leqq j\leqq N$, are all distinct positive integers, it follows from (3.1) that

$$\int_0^1 |S_n(x)|^4\,dx = R_n^2 + 2\sum_{j=1}^{N}(L_j^2+M_j^2)$$

and the lemma follows in virtue of (1.7).

PROOF OF THEOREM 1. By (1.6) and (1.7) we have

$$\begin{aligned}L+M &= \sum_{j=1}^{N}\left\{\sum_{m_l-m_k=K_j;1\leqq k<l\leqq n}\mathrm{Re}(c_k\bar{c}_l)\right\}^2\\ &\quad+\sum_{j=1}^{N}\left\{\sum_{m_l-m_k=K_j;1\leqq k<l\leqq n}\mathrm{Im}(c_k\bar{c}_l)\right\}^2\\ &\leqq \sum_{j=1}^{N}\left\{\sum_{m_l-m_k=K_j;1\leqq k<l\leqq n}|\mathrm{Re}(c_k\bar{c}_l)|\right\}^2\\ &\quad+\sum_{j=1}^{N}\left\{\sum_{m_l-m_k=K_j;1\leqq k<l\leqq n}|\mathrm{Im}(c_k\bar{c}_l)|\right\}^2\end{aligned}$$

and this by the Hölder inequality is

$$\leqq \sum_{j=1}^{N} r_j \sum_{m_l-m_k=K_j;1\leqq k<l\leqq n} |\mathrm{Re}(c_k\bar{c}_l)|^2$$
$$+\sum_{j=1}^{N} r_j \sum_{m_l-m_k=K_j;1\leqq k<l\leqq n} |\mathrm{Im}(c_k\bar{c}_l)|^2$$
$$\leqq r(n)\sum_{j=1}^{N} \sum_{m_l-m_k=K_j;1\leqq k<l\leqq n} \{|\mathrm{Re}(c_k\bar{c}_l)|^2+|\mathrm{Im}(c_k\bar{c}_l)|^2\}$$
$$(3.2)\qquad = r(n)\sum_{1\leqq k<l\leqq n} |c_k|^2\,|c_l|^2.$$

$$R_n^2 - T_n = \left\{\sum_{k=1}^{n}|c_k|^2\right\}^2 - \sum_{k=l}^{n}|c_k|^4 = 2\sum_{1\leqq k<l\leqq n}|c_k|^2\,|c_l|^2,$$

and hence, we have, by Lemma 1, (3.2), and the fact that $r(n)\geqq 1$,

$$(3.3)\qquad \int_0^1 |S_n(x)|^4\,dx \leqq R_n^2 + r(n)\{R_n^2 - T_n\}$$
$$\leqq R_n^2\{1+r(n)\} - T_n.$$

Now, by the Hölder inequality,

$$\left\{\int_0^1 |S_n(x)|\,dx\right\}^{2/3} \geqq \frac{\int_0^1 |S_n(x)|^2\,dx}{\left\{\int_0^1 |S_n(x)|^4\,dx\right\}^{1/3}}$$

and this, using the fact that

$$\int_0^1 |S_n(x)|^2\,dx = R_n \quad \text{(see (3.1)), and (3.3),}$$

$$\geqq \frac{R_n}{\{R_n^2(1+r(n)) - T_n\}^{1/3}}.$$

Theorem 1 is clear since by the Schwarz inequality $R_n^2\leqq nT_n$.

PROOF OF THEOREM 2. If

$$U_n(x) = \sum_{k=1}^{n}\gamma_k e(m_k x+\alpha_k) = \sum_{k=1}^{n}\gamma_k e(\alpha_k)e(m_k x),$$

$T_n(x)=\mathrm{Re}(U_n(x))$, and hence, using (3.3),

$$(3.4)\qquad \int_0^1 |T_n(x)|^4\,dx \leqq \int_0^1 |U_n(x)|^4\,dx \leqq \left\{\sum_{k=1}^{n}|\gamma_k e(\alpha_k)|^2\right\}^2(1+r(n))$$
$$-\sum_{k=1}^{n}|\gamma_k e(\alpha_k)|^4 = \left\{\sum_{k=1}^{n}\gamma_k^2\right\}^2(1+r(n)) - \sum_{k=1}^{n}\gamma_k^4.$$

From now on, the proof runs as that of Theorem 2 of [3] and we omit the details.

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