PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 36, Number 1, November 1972

CIRCLE ACTIONS ON HOMOTOPY SPHERES BOUNDING PLUMBING MANIFOLDS

REINHARD SCHULTZ[1]

ABSTRACT. Smooth circle actions are constructed on certain homotopy spheres not previously known to admit such actions.

In this paper we shall prove the following two results:

PROPOSITION A. *Let Σ^8 be any homotopy 8-sphere. Then there is a smooth semifree circle action on Σ^8 with S^4 as its fixed point set.*

PROPOSITION B. *Let Σ^{10} be any homotopy 10-sphere bounding a spin manifold. Then there is a smooth semifree circle action on Σ^{10} with S^4 as its fixed point set.*

Combining these with other results, we know that *any smooth manifold Σ^n which is piecewise-differentiably homeomorphic to S^n, bounds a spin manifold, and satisfies $n \leqq 13$ has a smooth circle action.* The above propositions imply the cases $n=8$, 10, while the cases $n=7$, 11, 12 follow because $\Gamma_n = \partial P_{n+1}$ in these cases and every homotopy sphere in ∂P_{n+1} ($n \geqq 5$) has a semifree circle action with a homotopy $(n-4)$-sphere as its fixed point set (e.g., see [3]). Finally, the cases $n=9$, 13 follow from the above remark on ∂P_{n+1} and results of Bredon [1].

Undoubtedly, the central difficulty in obtaining connected Lie group actions on homotopy spheres is the lack of a manageable construction for an arbitrary such manifold. The value of such a realization is obvious in the construction of large orthogonal group actions on homotopy spheres bounding π-manifolds. Bredon's construction of smooth S^1 and S^3 actions on homotopy spheres in the image of the Milnor-Munkres-Novikov pairing [1] is another illustration of the usefulness of an explicit construction for a given homotopy sphere. In this paper we shall show that certain homotopy spheres in the image of the Milnor plumbing pairing $\sigma_{p,q}: \pi_q(SO_p) \times \pi_p(SO_q) \to \Gamma_{p+q+1}$ (see [4] or [5]) also have smooth circle

Received by the editors July 26, 1971 and, in revised form, March 24, 1972.

AMS 1970 *subject classifications*. Primary 57D60, 57E15, 57E25; Secondary 57D50, 57D55.

Key words and phrases. Homotopy sphere, smooth semifree circle action, plumbing pairing, equivariant diffeomorphism, Puppe sequence, k-invariant, localized space.

[1] The author was partially supported by NSF grant GP-19530.

actions. Propositions A and B follow from this, results of D. Frank [**4**] stating that the image of $\sigma_{3,4}$ is Γ_8 and the image of $\sigma_{3,6}$ is $2\Gamma_{10}$, and some homotopy-theoretic computations. I am indebted to D. Frank for suggesting that results like Propositions A and B might be obtainable from [**4**].

1. **Constructing semifree circle actions.** We shall construct smooth semifree circle actions on certain homotopy spheres in the image of $\sigma_{p,q}$ by the familiar technique of pasting two S^1 manifolds together via an equivariant diffeomorphism. However, we shall be pasting along a semifree S^1 manifold rather than a free S^1 manifold (which is the usual way of studying semifree actions; e.g., see [**2**, Chapter III]).

Let p, q and r be positive integers satisfying $2r \leqq q+1$. Then S^1 acts orthogonally on R^{q+1} via the representation containing r copies of the standard 2-dimensional representation and a trivial representation; call this representation A_r. Then the induced smooth action on $S^p \times S^q$ given by

$$z \cdot (x, y) = (x, A_r(z)y), \qquad x \in S^p, y \in S^q \tag{1.1}$$

is semifree, and its fixed point set is $S^p \times S^{q-2r}$.

The orthogonal action A_r on S^q may be readily interpreted in terms of the homeomorphism from S^q to the join $S^{2r-1} * S^{q-2r}$; namely, it is the free linear action in the first join coordinate (an element of S^{2r-1}) and the trivial action on the other two join coordinates (elements of S^{q-2s} and $[0, 1]$ respectively). One advantage of this interpretation is that the orbit space projection is immediately recognized as the join of the canonical map $p_{r-1}: S^{2r-1} \to CP^{r-1}$ with the identity on S^{q-2r}. It is well known that this map is homotopically equivalent to the $(q-2r+1)$-fold suspension of p_{r-1}.

LEMMA 1.2. *Let $\alpha \in \pi_p(SO_q)$ be in the image of $\pi_p(U_r)$ and let $\beta \in \pi_q(SO_p)$ be in the image of*

$$(S^{q-2r+1}p_{r-1})^*: [S^{q-2r+1}CP^{r-1}, SO_p] \to \pi_q(SO_p).$$

Then the homotopy $(p+q+1)$-sphere corresponding to $\sigma_{p,q}(\alpha, \beta)$ has a semifree circle action whose fixed point set is an ordinary $(p+q+1-2r)$-sphere.

PROOF. Let $\varphi: S^p \to U_r$ be a smooth map representing α and let f be the diffeomorphism of $S^p \times S^q$ defined by $f(x, y) = (x, \varphi(x)y)$, where U_r acts orthogonally on S^q; then f is equivariant with respect to the S^1 action defined in (1.1). Let $\psi: S^q \to SO_p$ be a continuous map representing β; the assumption that β factors through $S^{q-2r+1}CP^{r-1}$ implies that ψ may be assumed to be constant on the orbits of the action A_r. If SO_p is taken with

the trivial S^1 action, then the smooth equivariant approximation theorem [7, 1.12] implies that ψ may also be assumed to be smooth. Hence the diffeomorphism $g(x, y)=(\psi(y)x, y)$ is also equivariant with respect to the S^1 action of (1.1); therefore, the commutator $[f, g]$ is also S^1 equivariant.

The action given in (1.1) obviously extends to smooth S^1 actions on the manifolds $D^{p+1}\times S^q$ and $S^p\times D^{q+1}$. Thus if these two manifolds with boundary are pasted together via the equivariant diffeomorphism $[f, g]$, the resulting closed manifold has a smooth circle action. But the resulting closed manifold is merely the homotopy sphere corresponding to $\sigma(\alpha, \beta)$; e.g., see [5, remark preceding 2.1]. Since the fixed point sets of the extended actions on $D^{p+1}\times S^q$ and $S^p\times D^{q+1}$ are $D^{p+1}\times S^{q-2r}$ and $S^p\times S^{q-2r}$ respectively, the fact that $[f, g]$ is the identity on $S^p\times S^{q-2r}$ implies that the fixed point set of the action on $\sigma(\alpha, \beta)$ is the ordinary sphere

$$S^{p+q+2r+1} = D^{p+1} \times S^{q-2r} \cup S^p \times D^{q-2r+1}.$$

It follows immediately that the circle action constructed above is semifree.

2. **Proof of main results.** The proofs of Propositions A and B are roughly parallel, although the latter is somewhat more complicated.

PROOF OF PROPOSITION A. Let $\alpha\in\pi_3(SO_4)$ map to the generator of $\pi_3(SO)=Z$, let β generate $\pi_3(SO_3)=Z$, and let η generate $\pi_4(S^3)=Z_2$. According to [4, Example 1, p. 565], the element $\sigma_{3,4}(\alpha, \beta\eta)\in\Gamma_8$ is nonzero for any choice of α. On the other hand, it is well known that α may be chosen to lie in the image of $\pi_3(U_2)$. Since η is the orbit space projection $S^0 * S^3\to S^0 * S^2$ described in §1, Proposition A follows immediately from Lemma 1.2.

The proof of Proposition B depends on homotopy computations involving 3-primary components; it will be convenient to localize all spaces at the prime 3. A discussion of localization functors appears in [6, Chapter II]; the main property we need is that the topological localization map $l_X: X\to X_{(p)}$ is given in homotopy by the algebraic localization $\pi_*(X)\to\pi_*(X)\otimes Z_{(p)}$. The following generalization is well known.

LEMMA 2.1. *Let X be a finite CW complex and let Y be a topological group, so that $[SX, Y]$ is naturally an abelian group. Then there is a natural isomorphism $\alpha_X: [SX, Y_{(p)}]\to[SX, Y]\otimes Z_{(p)}$ such that $\alpha_X l_{X*}$ is the canonical map $[SX, Y]\to[SX, Y]\otimes Z_{(p)}$.*

We shall need the following computational result:

LEMMA 2.2. *Let v' generate $\pi_6(S^3_{(3)})=Z_3$, and let $g: S^5\to CP^2$ be the canonical projection. Then v' is in the image of the map*

$$(Sg)^*: [SCP^2, S^3_{(3)}] \to \pi_6(S^3_{(3)}).$$

PROOF. Recall that the first two nonzero homotopy groups of $S^3_{(3)}$ are $Z_{(3)}$ in the dimension 3 and Z_3 in dimension 6. These are related by the nontrivial k-invariant $P^1 \in H^7(Z_{(3)}, 3; Z_3) = Z_3$.

By the exactness of the Puppe sequence and the mapping cone sequence

$$S^5 \xrightarrow{g} CP^2 \longrightarrow CP^3 \xrightarrow{h} S^6,$$

it suffices to show that $h^*\colon \pi_6(S^3_{(3)}) \to [CP^3, S^3_{(3)}]$ is the trivial map. But this is a straightforward consequence of the following two facts:

(i) P^1 is the first k-invariant of $S^3_{(3)}$.

(ii) The suspension of P^1 (i.e., $\iota^3 \in H^6(Z_{(3)}, 2)$) has a nontrivial image in $H^6(CP^3; Z_3)$.

REMARK. Since S^3 is the universal covering group of SO_3 and $\pi_1(SO_3) = Z_2$, there is an obvious homotopy equivalence from $S^3_{(3)}$ to $SO_{3(3)}$; consequently, the lemma remains true if S^3 is replaced by SO_3.

PROOF OF PROPOSITION B. Let $\alpha \in \pi_3(SO_6) = Z$ be a generator, and let β generate the 3-primary component of $\pi_6(SO_3) = Z_{12}$. As in the proof of Proposition A, we know that α is in the image of $\pi_3(U_3)$. On the other hand, Lemma 2.2 implies that β is in the image of $(Sg)^*\colon [SCP^2, SO_3] \to \pi_6(SO_3)$. Since $\sigma_{3,6}(\alpha, \beta)$ is nonzero by results of Frank [**4**, Example 2, p. 565], some exotic 10-sphere bounding a Spin manifold has a circle action of the desired type by Lemma 1.2. Since $b\mathrm{Spin}_{11} = Z_3$ is cyclic, an action on the other one may be constructed by taking an equivariant connected sum of this action with itself along the fixed point set.

REFERENCES

1. G. E. Bredon, *A π_*-module structure for Θ_* and applications to transformation groups*, Ann. of Math. (2) **86** (1967), 434–448. MR **36** #4570.

2. W. Browder, *Surgery and the theory of differentiable transformation groups*, Proc. Conf. on Transformation Groups (New Orleans, La., 1967), Springer, New York, 1968, pp. 1–46. MR **41** #6242.

3. W. Browder and T. Petrie, *Semi-free and quasi-free S^1 actions on homotopy spheres*, Essays on Topology and Related Topics (Mém. dédiés à Georges deRham), Springer, New York, 1970, pp. 136–146. MR **41** #7716.

4. D. L. Frank, *An invariant for almost-closed manifolds*, Bull. Amer. Math. Soc. **74** (1968), 562–567. MR **36** #5956.

5. R. Schultz, *Composition constructions on diffeomorphisms of $S^p \times S^q$*, Pacific J. Math. (to appear).

6. D. Sullivan, *Geometric topology*. I: *Localization, periodicity, and Galois symmetry*, M.I.T. Cambridge, Mass., 1970 (mimeographed).

7. A. G. Wasserman, *Equivariant differential topology*, Topology **8** (1969), 127–150. MR **40** #3563.

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907