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INTERSECTION THEORY IN AN EQUICHARACTERISTIC REGULAR LOCAL RING AND THE RELATIVE INTERSECTION THEORY

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ABSTRACT. Using the intersection theory of the notes of Serre, *Algèbre locale. Multiplicités*, the valuation theoretic formula for a hypersurface is given, and it is shown that transversality is equivalent to intersection multiplicity one. The intersection multiplicity of *Algèbre locale. Multiplicités* is computed for two algebraic varieties over an arbitrary field and compared to the intersection number of Weil's *Foundations of algebraic geometry*.

The intersection theory and the notation developed in the notes of Serre, *Algèbre locale. Multiplicités* [3], will be used. Let A be a regular noetherian ring which is locally equicharacteristic. Let Z be the group of cycles of A, the free abelian group generated by the prime ideals of A. If p is a prime ideal of A, p also denotes the cycle. If M is a noetherian A module and if $p_1, \cdots, p_s$ are the isolated prime ideals of M, let $Z(M)=\sum_{i=1,\cdots,s} \ell_{A_{p_i}}(M)p_i \in Z$. If p and q are two prime ideals of A which *intersect properly* at another prime ideal m of A, that is $p+q\subset m$ and height pA_m+ height $qA_m=$height $(p+q)A_m=$height mA_m, define as in [3, Chapter V], $i(p\cdot q, m)=\chi^{A_m}(A_m/pA_m, A_m/qA_m)$. Extending i bilinearly to cycles which intersect properly, $i(Z(M)\cdot Z(N), m)=\chi^{A_m}(M_m, N_m)$.

Let $p_1, \cdots, p_s$ be distinct prime ideals of A, let $q_1, \cdots, q_t$ be distinct prime ideals of A, and let a_i and b_j be nonzero integers for $i=1, \cdots, s$ and $j=1, \cdots, t$. The cycles $a_1p_1+\cdots+a_sp_s$ and $b_1q_1+\cdots+b_tq_t$ *intersect properly* if for each pair i, j the prime ideals p_i and q_j intersect properly at each isolated prime ideal m_{ijk} of p_i+q_j. If this holds, define

$$(a_1p_1 + \cdots + a_sp_s)\cdot(b_1q_1 + \cdots + b_tq_t) = \sum_{ijk} a_ib_ji(p_i\cdot q_j, m_{ijk})m_{ijk}$$

which is again a cycle. Associativity and the projection formula are given in [3, Chapter V, C].

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Below are given the valuation theoretic formula for a hypersurface section of a cycle and the equivalence of transversality and intersection multiplicity one. Both follow easily from **[3]**. A relative intersection number is then defined using the intersection theory of Weil's *Foundations of algebraic geometry* **[4]**, and it is shown that this intersection number is the intersection multiplicity given by the above theory.

1. **The valuation theoretic formula for the intersection of a cycle with a principal cycle.** Let A be an equicharacteristic regular local ring with maximal ideal m, and let p be a prime ideal of A such that the integral closure of A/p in its quotient field, $I(A/p)$, is a finite A/p module. Let $I(A/p)=(A/p)\alpha_1+\cdots+(A/p)\alpha_n$. Let $R=A[X_1,\cdots,X_n]$, the polynomial ring, which is regular. Let p' be the kernel of the homomorphism $h:R\to I(A/p)$ where $h|_A$ is the canonical homomorphism onto A/p and $h(X_i)=\alpha_i$. The projection $g:\operatorname{Spec} R\to\operatorname{Spec} A$ induced by $A\subset R$ is proper. Let $m_1,\cdots,m_s$ be the prime ideals of R lying over m which contain p'.

A *principal cycle* of A is a cycle of the form $Z(A/(f))$ where f is a nonzero element of A. Assume that the prime ideal p and $Z(A/(f))$ intersect properly.

By [**3**, p. V-29]

$$g_*(Z(R/p')\cdot g^*Z(A/(f))) = g_*Z(R/p')\cdot Z(A/(f)).$$

Now, $g^*Z(A/(f))=Z(R/(f))$, $g_*Z(R/p')=[Q(R/p'):Q(A/p)]p=p$, and $g_*m_i=[R/m_i:A/m]m$ where QS denotes the quotient field of an integral domain S. By [**3**, p. V-20, Corollary],

$$\chi^{R_{m_i}}(R_{m_i}/p'R_{m_i}, R_{m_i}/(f)) = \ell_{R_{m_i}}((R_{m_i}/p'R_{m_i})\otimes_{R_{m_i}}(R_{m_i}/(f)) = v_i(f)$$

where v_i is the order function of $R_{m_i}/p'R_{m_i}$, the valuation of R/p' centered at m_i/p'. Hence $p\cdot Z(A/(f))=g_*(\sum_i v_i(f)Z(R/m_i))$ and the desired formula follows,

$$i(p\cdot Z(A/(f)), m) = \sum_i [R/m_i:A/m]v_i(f).$$

2. **The equivalence of transversality and of intersection multiplicity one.** Let A be an equicharacteristic regular local ring, let m be its maximal ideal, and let k be the residue field A/m. If p is a prime ideal of A let $J(p)$ be the k-vector subspace $(p+m^2)/m^2$ of m/m^2. Two prime ideals p and q of A are transversal if and only if $\dim_k J(p+q)=\text{height } p+\text{height } q$.

PROPOSITION 1. *Let p and q be two prime ideals of A which intersect properly at m. Then $i(p\cdot q, m)=1$ if and only if p and q are transversal.*

PROOF. First let p and q be transversal. Then A/p, A/q and $A/(p+q)$ are regular local rings, A/p and A/q are Cohen-Macaulay, and $p+q=m$. By [**3**, Corollary, p. V-20]

$$\chi^A(A/p, A/q) = \ell_A(A/p \otimes_A A/q) = \ell_A(A/m) = 1.$$

Secondly let $i(p \cdot q, m)=1$. By [**3**, p. V-20]

$$1 = \chi^A(A/p, A/q) \geqq \ell_A(A/p \otimes_A A/q) \geqq 1.$$

So $\ell_A(A/p \otimes_A A/q)=1$, but $A/(p+q)$ is artinian, and hence $p+q=m$. Thus p and q are transversal, for A is regular.

3. **Intersection theory in algebraic geometry relative to an arbitrary field.** Let Ω be an algebraically closed field, let A_n denote the n-dimensional affine space with coordinates in Ω (the set of all n-tuples of elements of Ω), let k be a subfield of Ω, and let k_a be the algebraic closure of k in Ω. Let U and V be two k-varieties in A_n which intersect properly at a k-variety W. Let $U=\bigcup U_i$ and $V=\bigcup V_j$ where U_i and V_j are the distinct k_a-components of U and V respectively. Let W_1 be a k_a-component of W. Define an intersection multiplicity relative to k, by

$$i(U \cdot V, W) = \frac{[k(U):k]_\iota [k(V):k]_\iota}{[k(W):k]_\iota} \sum_{ij} i(U_i \cdot V_j, W_1)$$

where $i(U_i \cdot V_j, W_1)$ is the intersection number of Weil's *Foundations* [**4**, p. 148], and $[K:k]_\iota$ is the degree of inseparability of K over k.

If a k-variety V is considered to be a k-cycle on A_n, then V is of the form $V=[k(V):k]_\iota(V_1+\cdots+V_s)$ where $V_1, \cdots, V_s$ are the distinct k_a-components of V, and the above intersection multiplicity appears to be correct. Using the language and results of [**4**], associativity and the projection formula can be proven. Using the mixed jacobian of [**5**] for the definition of transversality, the equivalence of intersection multiplicity one and of transversality can be proven in this same language.

However it is more expedient to use the intersection theory of [**3**].

PROPOSITION 2. *Let U and V be two varieties in A_n intersecting properly at a k-variety W. Let $\mathcal{O}$ be the local ring of A_n at W over k ($\mathcal{O}= \{F/G \mid F, G \in k[A_n]=k[X_1, \cdots, X_n], G(W)\neq 0\}$), let m be its maximal ideal, and let p and q be the prime ideals in $\mathcal{O}$ of U and of V respectively. Then,*

$$i(U \cdot V, W) = i(p \cdot q, m).$$

PROOF. Let k' be a finite field extension of k over which the k_a-components of U, V and W are regular. Let $\mathcal{O}'$ be $\mathcal{O} \otimes_k k'$ reduced. The map

f: Spec $\mathcal{O}' \to$ Spec $\mathcal{O}$ induced by the inclusion is proper, and by [3, p. V-29], $f_*(f^*p \cdot f^*q) = (f_*f^*p) \cdot q$. Let $p_1, \cdots, p_s$ and $q_1, \cdots, q_t$ be the prime ideals of $\mathcal{O}'$ lying over p and q respectively. Let $m_1, \cdots, m_r$ be the prime ideals of $\mathcal{O}'$ lying over m.

Let $\mathcal{O}_i$ denote the localization of $\mathcal{O} \otimes_k k'$ at p_i, and let p_i also denote the maximal ideal of $\mathcal{O}_i$. Then

$$f^*p = \sum_{i=1,\cdots,s} \ell_{\mathcal{O}'_{p_i}}(\mathcal{O}/p \otimes_{\mathcal{O}} \mathcal{O}'_{p_i}) \cdot p_i$$

and

$$\ell_{\mathcal{O}'_{p_i}}(\mathcal{O}/p \otimes_{\mathcal{O}} \mathcal{O}'_{p_i}) = \ell_{\mathcal{O}_i}(\mathcal{O}/p \otimes_{\mathcal{O}} \mathcal{O}_i) = \ell_{\mathcal{O}_i}(K \otimes_k k')$$

where $K = \mathcal{O}_p/p\mathcal{O}_p = k(U)$. Let L be a purely transcendental extension of k with K (finite) algebraic over L. L and k' are linearly disjoint over k, so $L \otimes_k k' = Lk'$ and $K \otimes_k k' = K \otimes_L Lk'$. $K \otimes_L Lk'$ is artinian, the prime ideals $\mathfrak{p}_i$ of $K \otimes_L Lk'$ correspond to the p_i, and $\mathcal{O}_i$ acts on $(K \otimes_L Lk')_{\mathfrak{p}_i}$ which is the component of $K \otimes_L Lk'$ not contained in $\mathfrak{p}_i$. The p_i are conjugate over k, the $\mathfrak{p}_i$ are also conjugate over k, and $(K \otimes_L Lk')/\mathfrak{p}_i \simeq k'(U_i) \simeq Kk'$.

$$\begin{aligned} \ell_{\mathcal{O}_i}(K \otimes_L Lk') &= s^{-1}\ell_{Kk'}(K \otimes_L Lk') = s^{-1}[Kk':L]^{-1} \dim_L(K \otimes_L Lk') \\ &= s^{-1}[Kk':L]^{-1}[k':k][K:L] \\ &= s^{-1}[k'(U_i):k(U)]^{-1}[k':k] = [k(U):k]_\iota, \end{aligned}$$

for $s = [k':k]_s/[k'(U_i):k(U)]_s$ and $[k(U):k]_\iota = [k':k]_i/[k'(U_i):k(U)]_i$ by [4, Propositions 26, 27, p. 23]. Thus $f^*p = [k(U):k]_\iota(p_1 + \cdots + p_s)$. Now, $f_*m_i = \ell_{\mathcal{O}}(\mathcal{O}'_{m_i}/m_i\mathcal{O}'_{m_i})m$, and

$$\ell_{\mathcal{O}}(\mathcal{O}'_{m_i}/m_i\mathcal{O}'_{m_i}) = \ell_{\mathcal{O}/m}(k'(W_1)) = [k'(W_1):k(W)] = [k:k']/r[k(W):k]_\iota.$$

By the first computation, and a computation similar to the second, $f_*f^*p = [k':k]p$. Thus,

$$[k':k]i(p \cdot q, m) = \frac{[k:k'][k(U):k]_\iota[k(V):k]_\iota}{r[k(W):k]_\iota} \sum_{ijk} i(p_i \cdot q_j, m_k)$$

and

$$i(p \cdot q, m) = \frac{[k(U):k]_\iota[k(V):k]_\iota}{[k(W):k]_\iota} \sum_{ij} i(p_i \cdot q_j, m_1).$$

If k'' is a finite algebraic extension of k' it follows, by the above, that $\chi^{\mathcal{O}'}(\mathcal{O}'/p_i, \mathcal{O}'/q_j) = \chi^{\mathcal{O}''}(\mathcal{O}''/p'', \mathcal{O}''/q'')$, where $\mathcal{O}'' = \mathcal{O}' \otimes_k k''$, $p'' = p_i \otimes_k k''$, $q'' = q_j \otimes_k k''$. $\chi^A(M, N) = \sum_i \ell_A \operatorname{Tor}_i^A(M, n)$ commutes with direct limits, so $\chi^{\mathcal{O}'}(\mathcal{O}'/p_i, \mathcal{O}'/q_j) = i(U_i \cdot V_j, W_1)$ by [3, p. V-21, Theorem 1].

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