PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 36, Number 1, November 1972

A NOTE ON INTEGRAL CLOSURE

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ABSTRACT. Let R be an integrally closed domain and x_i, y_j $(1 \leqq i \leqq n, 1 \leqq j \leqq m)$ R-sequences. Let

$$T = R[x_1^{\alpha_1} \cdots x_n^{\alpha_n}/y_1^{\beta_1} \cdots y_m^{\beta_m}],$$

where the α_i and β_j are positive integers. If T is integrally closed then

$$(*) \qquad \alpha_1 = \cdots = \alpha_n = 1 \quad \text{or} \quad \beta_1 = \cdots = \beta_m = 1.$$

$(*)$ is sufficient for T to be integrally closed in the following cases:
(1) R is Noetherian and the $(x_i, y_j)R$ are distinct prime ideals,
(2) R is a polynomial ring over an integrally closed domain and the x_i and y_j are indeterminates.

It is known ([2], [3]) that the monoidal transform of a domain R with respect to an ideal I is normal (i.e., integrally closed) if high powers of I are complete, and that the converse holds provided that R is Noetherian. However, in most instances, the criterion of completeness is not very practical for proving integral closure. This paper is concerned with the simplest case, namely the preservation of integral closure of a domain R upon adjunction of a quotient a/b of certain elements of R. The condition on a, b is symmetric so this work may alternately be viewed as an investigation of the completeness of the ideals $(a, b)^k$ for all large k. I would like to thank the referee for a number of helpful suggestions.

The following notation will be fixed throughout. Let R be an integrally closed domain. Let x_i, y_j $(1 \leqq i \leqq n, 1 \leqq j \leqq m)$ be R-sequences and let

$$T = R[x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}/y_1^{\beta_1} y_2^{\beta_2} \cdots y_m^{\beta_m}],$$

where the α_i and β_j are positive integers.

PROPOSITION 1. *If T is integrally closed, then*

$$(*) \qquad \alpha_1 = \alpha_2 = \cdots = \alpha_n = 1 \quad \text{or} \quad \beta_1 = \beta_2 = \cdots = \beta_m = 1.$$

PROOF. Assume $\alpha_i > 1$ and $\beta_j > 1$ for some i and j. Then

$$(x_1^{\alpha_1} \cdots x_i^{\alpha_i - 1} \cdots x_n^{\alpha_n}/y_j)^2 = (x_1^{\alpha_1} \cdots x_i^{\alpha_i - 2} \cdots x_n^{\alpha_n})(x_1^{\alpha_1} \cdots x_i^{\alpha_i} \cdots x_n^{\alpha_n}/y_j^2)$$

Received by the editors September 17, 1971 and, in revised form, January 26, 1972.
AMS 1970 *subject classifications*. Primary 13B20, 13E05, 14A05.
Key words and phrases. Integrally closed domain, Noetherian domain, R-sequence, monoidal transform, complete ideal.

is an element of T. To contradict the fact that T is integrally closed we must show that $x_1^{\alpha_1}\cdots x_i^{\alpha_i-1}\cdots x_n^{\alpha_n}/y_j$ is not in T. Suppose it is. Then we have

$$x_1^{\alpha_1}\cdots x_i^{\alpha_i-1}\cdots x_n^{\alpha_n}/y_j = r_0 + r_1(x_1^{\alpha_1}\cdots x_n^{\alpha_n}/y_1^{\beta_1}\cdots y_m^{\beta_m}) + \cdots + r_k(x_1^{\alpha_1}\cdots x_n^{\alpha_n}/y_1^{\beta_1}\cdots y_m^{\beta_m})^k,$$

where the r_q are in R and $k\geqq 1$. Thus,

$$y_1^{k\beta_1}\cdots y_j^{k\beta_j-1}\cdots y_m^{k\beta_m}(x_1^{\alpha_1}\cdots x_i^{\alpha_i-1}\cdots x_n^{\alpha_n}) = r_0(y_1^{\beta_1}\cdots y_m^{\beta_m})^k + r_1(x_1^{\alpha_1}\cdots x_n^{\alpha_n})(y_1^{\beta_1}\cdots y_m^{\beta_m})^{k-1} + \cdots + r_k(x_1^{\alpha_1}\cdots x_n^{\alpha_n})^k.$$

It follows that $r_0 = r_0'(x_1^{\alpha_1}\cdots x_i^{\alpha_i-1}\cdots x_n^{\alpha_n})$, for some $r_0'\in R$. Hence,

$$(1-r_0'y_j)(y_1^{k\beta_1}\cdots y_j^{k\beta_j-1}\cdots y_m^{k\beta_m}) = x_i(r_1y_1^{\beta_1}\cdots y_m^{\beta_m} + \cdots + r_k(x_1^{\alpha_1}\cdots x_n^{\alpha_n})^{k-1}).$$

This gives the contradiction $1\in(x_i, y_j)R$. $\square$

Note that the adjunction of a quotient x_1/y_1 of irreducible R-sequence elements is not sufficient for the integral closure of $R[x_1/y_1]$. (For example, let K be a field, x, y indeterminates and $R=K[x, y, x^2/y]_P$, where $P=(x, y, x^2/y)$. If $x_1=x^2/y$ and $y_1=y$, then $R[x_1/y_1]$ is not integrally closed.) In the Noetherian case, with an additional hypothesis on the ideals $(x_i, y_j)R$, we have a converse to Proposition 1.

THEOREM 2. *Assume in addition that R is Noetherian and that the ideals $(x_i, y_j)R$ are distinct prime ideals. Then* (*) *is sufficient for T to be integrally closed.*

REMARKS. (i) If R is a polynomial ring over a field and the x_i and y_j are indeterminates, then the statement of the theorem is easily checked using the Jacobian criterion.

(ii) Under the further assumption that $x_i, y_1, \cdots, y_m$ and $y_j, x_1, \cdots, x_n$ $(1\leqq i\leqq n,\ 1\leqq j\leqq m)$ are R-sequences, R. Fossum has given a direct proof of the fact that the ideals

$$(x_1^{\alpha_1}\cdots x_n^{\alpha_n}, y_1\cdots y_m)^k, \qquad (x_1\cdots x_n, y_1^{\beta_1}\cdots y_m^{\beta_m})^k$$

are complete for all k. It then follows from ([**2**], [**3**]) that T is integrally closed.

PROOF OF THEOREM 2. Since any domain is the intersection of its localizations at maximal primes of principal ideals [**1**, Theorem 53], we will show that all such localizations of T are integrally closed. Let Q be a prime ideal of T. If $y_j\notin Q$ for all $1\leqq j\leqq m$, then, since $T\subset R[1/y_1^{\beta_1}\cdots y_m^{\beta_m}]$, we have that $T_Q=R_{Q\cap R}$ and T_Q is, therefore, integrally closed. Suppose that $y_j \in Q$ for some j. We use the fact that

$$T = R[t]/(y_1^{\beta_1} \cdots y_m^{\beta_m} t - x_1^{\alpha_1} \cdots x_n^{\alpha_n}),$$

where t is an indeterminate [**1**, p. 102, Exercise 3]. Let Q' denote the inverse image of Q in $R[t]$. Now $y_j \in Q$ implies that $(x_i, y_j)R \subseteq Q \cap R$ for some i. We distinguish two cases. First, assume that $Q' = (x_i, y_j)R[t]$ so that $Q = (x_i, y_j)T$. Note that none of the elements $x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n, y_1, \cdots, y_{j-1}, y_{j+1}, \cdots, y_m, x_1^{\alpha_1} \cdots x_n^{\alpha_n}/y_1^{\beta_1} \cdots y_m^{\beta_m}$ are in Q. If $\alpha_1 = \cdots = \alpha_n = 1$, then $QT_Q = y_j T_Q$. If some $\alpha_k > 1$ but $\beta_1 = \cdots = \beta_m = 1$, then $QT_Q = x_i T_Q$. Thus (*) implies that T_Q is a discrete valuation ring (DVR). (Note that if T is Macaulay, e.g. if R is Macaulay, the proof is finished since we have shown that T_Q is a DVR for all rank 1 primes.) To complete the proof, we consider the case $Q' \supsetneq (x_i, y_j)R[t]$. In this case $Q'/(x_i, y_j)R[t]$ is a prime ideal of rank $\geqq 1$ in the domain

$$R[t]/(x_i, y_j)R[t].$$

It follows that Q' contains an R-sequence of length 3. Thus Q contains an R-sequence of length 2 and cannot belong to a principal ideal. $\square$

If R is not Noetherian, one might drop down to a Noetherian subring R_0. However, in general, the ideals $(x_i, y_j)R_0$ will not be prime. This technique will work in the following case.

COROLLARY 3. *Let $R = S[x_1, \cdots, x_n, y_1, \cdots, y_m]$, where S is an integrally closed domain and the x_i and y_j are indeterminates. Then (*) is sufficient for T to be integrally closed.*

PROOF. Let $z = f/g$ with $f, g \in R$. Suppose that there is an equation $z^k + t_1 z^{k-1} + \cdots + t_k = 0$, where $t_i \in T$, $1 \leqq i \leqq k$. Each t_i is a polynomial in $x_1^{\alpha_1} \cdots x_n^{\alpha_n}/y_1^{\beta_1} \cdots y_m^{\beta_m}$ with coefficients $h_{i0}, \cdots, h_{id_i}$ in R. Let S_0 be the prime integral domain of S. Let S_1 be the ring generated over S_0 by the coefficients of f and g and the coefficients of all the h_{ij}, $1 \leqq i \leqq k$, $1 \leqq j \leqq d_i$. $S_1 \subseteq S$. Let $R_1 = S_1'[x_1, \cdots, x_n, y_1, \cdots, y_m]$, where S_1' is the integral closure of S_1 (in its quotient field). $S_1' \subseteq S$ so that the x_i and y_j are indeterminates over S_1'. R_1 is Noetherian since S_1' is [**4**, (37.5), (35.3)]. Now z is integral over $T_1 = R_1[x_1^{\alpha_1} \cdots x_n^{\alpha_n}/y_1^{\beta_1} \cdots y_m^{\beta_m}]$. By Theorem 2, $z \in T_1 \subseteq T$. Thus, T is integrally closed. $\square$

We conclude with a remark concerning the general case of a monoidal transform of a domain R with respect to an arbitrary ideal I. In [**5**] it is proved that I^k is complete for all k if the following conditions are satisfied: (1) R is integrally closed, (2) $\bigcap_{j=0}^{\infty} I^j = 0$, (3) $G_I(R)$, the associated graded ring of R with respect to I, is a domain. Actually, the proof uses (1), (2) and the fact that $G_I(R)$ contains no nilpotent elements so that (3) may be replaced by (3') $G_I(R)$ is reduced. Thus, for example, if R is an integrally closed domain, and I is a radical ideal which is generated by an R-sequence $a_1, \cdots, a_m$, and satisfies $\bigcap_{j=0}^{\infty} I^j = 0$, then I^k is complete for all k. By ([2], [3]) the monoidal transform of R with respect to I is normal.

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