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ON THE INDIVIDUAL ERGODIC THEOREM FOR POSITIVE OPERATORS

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ABSTRACT. A theorem which gives a condition on a positive linear contraction on an L^1-space in order that the individual ergodic theorem hold is proved. The theorem contains a result obtained by Y. Ito as a special case.

Let $(X, \mathscr{M}, m)$ be a σ-finite measure space and let T be a positive linear contraction on $L^1(m)$. Let $a_{n,j}$ be a matrix of numbers such that

$$\sum_{j=0}^{\infty} |a_{n,j}| < \infty \quad \text{for } n = 0, 1, \cdots, \tag{1}$$

$$\lim_{n'} \sum_{j=0}^{\infty} a_{n',j} = 1, \tag{2}$$

$$\lim_{n'} \sum_{j=0}^{\infty} a_{n',j} b_{j+1} = b \tag{3}$$

whenever $b_0, b_1, \cdots$ is a bounded sequence of numbers for which $\lim_{n'} \sum_{j=0}^{\infty} a_{n',j} b_j = b$ exists and is finite, where $\{n'\}$ is a subsequence of $\{n\}$.

Under these conditions we shall prove the following

THEOREM. *If there exists a strictly positive function h in $L^1(m)$ such that the set $\{\sum_{j=0}^{\infty} a_{n,j} T^j h;\ n \geqq 0\}$ is weakly sequentially compact in $L^1(m)$, then for any $f \in L^1(m)$ the limit*

$$\lim_{n} \frac{1}{n} \sum_{j=0}^{n-1} T^j f(x) \tag{4}$$

exists and is finite almost everywhere.

The following proof is a generalization of that given by Y. Ito in **[6]**.

PROOF. Let $g \in L^1(m)$ and $\{n'\}$ a subsequence of $\{n\}$ such that

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$\sum_{j=0}^{\infty} a_{n',j} T^j h$ converges weakly to g. Then for any $u \in L^\infty(m)$ we have

$$\begin{aligned}\int gu\, dm &= \lim_{n'} \sum_{j=0}^{\infty} a_{n',j} \int (T^j h) u\, dm \\ &= \lim_{n'} \sum_{j=0}^{\infty} a_{n',j} \int (T^{j+1} h) u\, dm = \int (Tg) u\, dm.\end{aligned}$$

This implies that $g = Tg$. Next suppose that $\int (f - Tf) u\, dm = 0$ for any $f \in L^1(m)$. Then, clearly, $\int (f - T^n f) u\, dm = 0$ for all $n \geqq 0$, and hence

$$\begin{aligned}\int gu\, dm &= \lim_{n'} \sum_{j=0}^{\infty} a_{n',j} \int (T^j h) u\, dm \\ &= \lim_{n'} \sum_{j=0}^{\infty} a_{n',j} \int hu\, dm = \int hu\, dm.\end{aligned}$$

It follows that $h - g$ belongs to the closed linear manifold generated by the set $\{f - Tf; f \in L^1(m)\}$. Thus

$$\lim_n \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^j h - g \right\|_1 = 0,$$

and hence $g \geqq 0$. Let $A = \{x \in X; g(x) = 0\}$, and let the conservative and dissipative parts [3] of T be C and D, respectively. We shall first prove that $A = D$. It is clear that $D \subset A$. To see that $A \subset D$, let T^* denote the corresponding adjoint operator on $L^\infty(m)$. Since $Tg = g$, it follows that $T^* 1_A \leqq 1_A$, whence if we define $B = A \cap C$ then $T^{*j} 1_B = 1_B$ almost everywhere on C for each $j \geqq 0$. Thus

$$\begin{aligned}\int h 1_B\, dm &\leqq \lim_n \int \frac{1}{n} \sum_{j=0}^{n-1} h T^{*j} 1_B\, dm \\ &= \lim_n \int \left(\frac{1}{n} \sum_{j=0}^{n-1} T^j h \right) 1_B\, dm = \int g 1_B\, dm = 0.\end{aligned}$$

Since h is strictly positive, it follows that $m(B) = 0$, and hence $A \subset D$.

Let f be any function in $L^1(m)$. Since $A = D$, it follows at once that the limit (4) exists and is finite almost everywhere on A. On the other hand, the Chacon-Ornstein theorem [4] implies that

$$\lim_n \frac{1}{n} \sum_{j=0}^{n-1} T^j f(x) = g(x) \lim_n \left(\sum_{j=0}^{n-1} T^j f(x) \Big/ \sum_{j=0}^{n-1} T^j g(x) \right)$$

exists and is finite almost everywhere on $X - A$. This completes the proof of the theorem.

It should be pointed out here that if $a_{n,j}$ is a regular matrix such that

$$\lim_{k\to\infty} \sum_{j=k}^{\infty} |a_{n,j+1} - a_{n,j}| = 0$$

uniformly in n then it satisfies (1), (2) and (3) (see [5]).

REMARK 1. Let $\{w_n; n \geqq 1\}$ be a sequence of nonnegative numbers whose sum is one, and let $\{u_n; n \geqq 0\}$ be the sequence defined by $u_n = w_1 u_{n-1} + \cdots + w_n u_0$, $u_0 = 1$. Then the above argument together with Baxter's ergodic theorem [2] implies that under the same condition as in the theorem, for any $f \in L^1(m)$ the limit

$$\lim_n \left(\sum_{j=0}^{n-1} u_j T^j f(x) \Big/ \sum_{j=0}^{n-1} u_j \right) \tag{5}$$

exists and is finite almost everywhere. The theorem is a special case of this result.

REMARK 2. If T maps, in addition, $L^p(m)$ into $L^p(m)$ and $\|T\|_p \leqq 1$ for some p with $p > 1$, then for any $f \in L^1(m)$ the limit (5) exists and is finite almost everywhere. This follows from [1] and [7].

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