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A VARIATIONAL PROBLEM FOR SUBMANIFOLDS OF EUCLIDEAN SPACE

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ABSTRACT. Let M^n be a compact differentiable manifold and R^{n+k} Euclidean space. A necessary and sufficient condition is given for an immersion $\psi: M^n \to R^{n+k}$ to be a stationary immersion for $J=\int_{M_\psi^n} \langle x-x_c, x-x_c\rangle\, dv$ subject to the side condition $V=\int_{M_\psi^n} dv=$ a fixed constant, where x_c is the center of mass. In particular, minimal submanifolds of spheres satisfy this condition.

1. **Introduction.** Let M^n be an n-dimensional compact differentiable manifold. An immersion $\psi: M^n \to R^{n+k}$ induces a Riemannian metric on M^n; M^n with this Riemannian metric is denoted by M_ψ^n. Let x denote the position vector in R^{n+k}, and let x_c denote the center of mass of M_ψ^n in R^{n+k}; i.e., $x_c=(1/V)\int_{M_\psi^n} x\, dv$, where $V=\int_{M_\psi^n} dv$ and dv is the volume element on M_ψ^n. For $p \in M_\psi^n$, the tangent space $T_p(M_\psi^n)$ is identified with a subspace of $T_{\psi(p)}(R^{n+k})$. The normal space $T_p^\perp(M_\psi^n)$ is the subspace of $T_{\psi(p)}(R^{n+k})$ consisting of all $X \in T_{\psi(p)}(R^{n+k})$ which are orthogonal to $T_p(M_\psi^n)$. For $q \in R^{n+k}$, $T_q(R^{n+k})$ is identified with $T_0(R^{n+k})$ by parallel translation, where 0 is the origin in R^{n+k}; and $T_0(R^{n+k})$ is identified with R^{n+k}. If $z: M^n \to R^{n+k}$, we consider z as a vector field defined along ψ by the above identifications. Let $z_N(p)$ be the orthogonal projection of $z(p)$ into $T_p^\perp(M_\psi^n)$ and z_T the orthogonal projection of $z(p)$ into $T_p(M_\psi^n)$. The Euclidean inner product will be denoted by $\langle\ ,\ \rangle$.

THEOREM. *The immersion $\varphi: M^n \to R^{n+k}$ is a stationary immersion for $J=\int_{M_\psi^n} \langle x-x_c, x-x_c\rangle\, dv$ subject to the side condition $V=\int_{M_\psi^n} dv=$ a fixed constant, say V_0, if and only if $(x-x_c)_N=\frac{1}{2}(\langle x-x_c, x-x_c\rangle+\lambda)\eta$ and $\int_{M_\psi^n} dv=V_0$, where λ is a constant and η is the mean curvature normal* [**2**, *p*. 34].[1]

The stationary character of φ means that if ψ_t, $t \in (-\varepsilon, \varepsilon)$, is any one parameter family of immersions with $\psi_0=\varphi$ and $V_0=\int_{M_{\psi t}^n} dv$ for all t, then $dJ(0)/dt=0$.

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[1] The definition of η in [2] differs from our usage by a factor of $1/n$.

LEMMA. *The immersion $\varphi: M^n \to R^{n+k}$ satisfies*

$$x - x_c = \tfrac{1}{2}(\langle x - x_c, x - x_c\rangle + \lambda)\eta,$$

λ a constant, if and only if φ immerses M^n as a minimal submanifold of a sphere S^{n+k-1}.

As an immediate consequence of the Theorem and the Lemma we have:

COROLLARY. *If $\varphi: M^n \to R^{n+k}$ immerses M^n as a minimal submanifold of a sphere S^{n+k-1} with $V_0 = \int_{M_\varphi^n} dv$, then φ is a stationary immersion for J subject to $V = V_0$.*

Brian Smyth has pointed out the following Proposition to me.

PROPOSITION. *If $\varphi: M^n \to R^{n+k}$ is a stationary immersion for J subject to V=constant and in addition $\langle \eta, \eta\rangle$ is constant on M^n, then φ immerses M^n as a minimal submanifold of a sphere.*

All immersions, vector fields, etc. are assumed to be C^∞.

2. **Proofs.**

PROOF OF THEOREM. Let ψ_t be a 1-parameter family of immersions of M^n into R^{n+k} with $\psi_0 = \varphi$. Let y denote the position vector in R^{n+k} for ψ_t and x the position vector for φ. Assume $x_c = \mathbf{0}$. Let $u = dy(0)/dt$. It is well known that (see [1, p. 74] for the case of a surface in R^3)

$$\frac{dV}{dt}(0) = -\int_{M_\varphi^n} \langle \eta, u\rangle\, dv + \int_{M_\varphi^n} \operatorname{Div} u_T\, dv \tag{1}$$
$$= -\int_{M_\varphi^n} \langle \eta, u\rangle\, dv.$$

For $dJ(0)/dt$ we have

$$\frac{dJ}{dt}(0) = \int_{M_\varphi^n} \langle x, x\rangle(\langle -\eta, u\rangle + \operatorname{Div} u_T)\, dv$$
$$+ \int_{M_\varphi^n} \left(\frac{d}{dt}\langle y - y_c, y - y_c\rangle\right)(0)\, dv.$$

Using $x_c = \mathbf{0}$ (and therefore $\int_{M_\varphi^n} \langle x, b\rangle\, dv = 0$ for a constant vector b) and

$$\begin{aligned}\langle x, x\rangle \operatorname{Div} u_T &= \operatorname{Div}\langle x, x\rangle u_T - \langle u_T, \operatorname{grad}\langle x, x\rangle\rangle \\ &= \operatorname{Div}\langle x, x\rangle u_T - 2\langle u_T, x\rangle \\ &= \operatorname{Div}\langle x, x\rangle u_T - 2\langle u, x_T\rangle,\end{aligned}$$

we easily find

$$\frac{dJ}{dt}(0) = \int_{M^n_\varphi} \langle 2\boldsymbol{x} - 2\boldsymbol{x}_T - \langle \boldsymbol{x}, \boldsymbol{x}\rangle \eta, \boldsymbol{u}\rangle \, dv \tag{2}$$
$$= \int_{M^n_\varphi} \langle 2\boldsymbol{x}_N - \langle \boldsymbol{x}, \boldsymbol{x}\rangle \eta, \boldsymbol{u}\rangle \, dv.$$

Since M^n is compact, η is not identically zero. Appealing to the well-known method of Euler-Lagrange multipliers for variational problems with side conditions, we conclude from (1) and (2) that a necessary and sufficient condition for φ to be a stationary immersion for J subject to V=constant is that there exist a constant λ (the Euler-Lagrange multiplier) such that $2\boldsymbol{x}_N - \langle \boldsymbol{x}, \boldsymbol{x}\rangle\eta = \lambda\eta$; i.e.,

$$2\boldsymbol{x}_N = (\langle \boldsymbol{x}, \boldsymbol{x}\rangle + \lambda)\eta. \tag{3}$$

Proof of Lemma. (i) Assume $\boldsymbol{x}_c = \boldsymbol{0}$. Suppose $\boldsymbol{x} = \frac{1}{2}(\langle \boldsymbol{x}, \boldsymbol{x}\rangle + \lambda)\eta$. Clearly $\boldsymbol{x}_N = \boldsymbol{x}$. Let X be tangent to M^n. Then $X\langle \boldsymbol{x}, \boldsymbol{x}\rangle = 2\langle X, \boldsymbol{x}\rangle = 0$, since $\boldsymbol{x}_N = \boldsymbol{x}$. Thus $\langle \boldsymbol{x}, \boldsymbol{x}\rangle$ is constant on M^n and $\boldsymbol{x}$=(constant) η. This implies that φ immerses M^n as a minimal submanifold of a sphere with center at the origin.

(ii) Suppose φ immerses M^n as a minimal submanifold of a sphere with center $\boldsymbol{a}$. Since $\boldsymbol{x} - \boldsymbol{a} = \mu\eta$ for some constant μ, it suffices to show that $\boldsymbol{x}_c = \boldsymbol{a}$. Let $f = \langle \boldsymbol{x}, \boldsymbol{b}\rangle$, where $\boldsymbol{b}$ is a constant vector. Then $\Delta f = \langle \eta, \boldsymbol{b}\rangle$, where Δ is the Laplacian on M^n (see [2, p. 340]). Hence, $\int_{M^n_\varphi} \langle \eta, \boldsymbol{b}\rangle \, dv = 0$; and thus $\int_{M^n_\varphi} \eta \, dv = \boldsymbol{0}$. But $\boldsymbol{x} - \boldsymbol{a} = \mu\eta$. Thus $\int_{M^n_\varphi} (\boldsymbol{x} - \boldsymbol{a}) \, dv = \boldsymbol{0}$; i.e., $\boldsymbol{x}_c = \boldsymbol{a}$.

Proof of Proposition. Assume $\boldsymbol{x}_c = \boldsymbol{0}$, and let $H = \langle \eta, \eta\rangle$. Let $f = \frac{1}{2}\langle \boldsymbol{x}, \boldsymbol{x}\rangle$. Then, it is not difficult to show that $\Delta f = n + \langle \boldsymbol{x}, \eta\rangle$, where Δ is the Laplacian on M^n. At a local maximum of f, we must have $\boldsymbol{x} = \boldsymbol{x}_N$ and $\Delta f \leqq 0$. Thus, $\langle \boldsymbol{x}, \eta\rangle \leqq -n$; and using (3) we obtain $f \leqq (-2n/H) - \lambda$ at a local maximum of f. Similarly, at a local minimum of f we have $f \geqq (-2n/H) - \lambda$. Thus, f is constant on M^n and $\boldsymbol{x}$=(constant) η. This implies that φ immerses M^n as a minimal submanifold of a sphere.

Remark 1. It would be interesting to know whether or not all solutions of the variational problem considered in this paper are minimal submanifolds of spheres.

Remark 2.

$$\iint_{M^n_\varphi \times M^n_\varphi} \langle \boldsymbol{x}_1 - \boldsymbol{x}_2, \boldsymbol{x}_1 - \boldsymbol{x}_2\rangle \, dv_1 \, dv_2 = 2V \int_{M^n_\varphi} \langle \boldsymbol{x} - \boldsymbol{x}_c, \boldsymbol{x} - \boldsymbol{x}_c\rangle \, dv,$$

where $\boldsymbol{x}_i$ is the position vector and dv_i the volume element for the ith factor of $M^n_\varphi \times M^n_\varphi$.

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