PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 36, No. 2, December 1972

CHARACTERIZING TOPOLOGIES BY FUNCTIONS

MELVIN C. THORNTON

ABSTRACT. The multiplicative structure of the idempotents in the semiring of nonnegative lower semicontinuous functions on a large class of spaces determines the topology of the space.

Nielsen and Sloyer [2] proved that compact T_1 spaces are homeomorphic iff their semirings of nonnegative lower semicontinuous functions are isomorphic. This note shows that the compact T_1 assumption is much too strong and that only an isomorphism between the idempotent elements is necessary. Let $L^+(X)$ denote the semiring of all nonnegative real-valued lower semicontinuous functions on X with multiplication and addition defined pointwise. A space is a T_D space iff the derived set of every point is closed [3].

THEOREM. *Let X and Y be T_D spaces. Then X is homeomorphic to Y iff the idempotents in $L^+(X)$ are isomorphic to the idempotents in $L^+(Y)$.*

PROOF. Let $U(X)\subset L^+(X)$ denote the set of idempotents. The elements of $U(X)$ are precisely the characteristic functions of the open sets in X. $U(X)$ is a Brouwerian lattice where $f\leqq g$ iff $f(x)\leqq g(x)$ for all x, $f\vee g(x)=\max\{f(x),g(x)\}$, and $f\wedge g(x)=\min\{f(x),g(x)\}$. These lattice operations are determined by the algebra of the semiring as follows. The ordering $f\leqq g$ holds iff $f\cdot g=f$. The meet is $f\wedge g=f\cdot g$. The join of f and g is that unique element $(f\vee g)\in U(X)$ such that $(f\vee g)\cdot f=f$, $(f\vee g)\cdot g=g$, and with the property that if $h\cdot f=f$, $h\cdot g=g$, then, $(f\vee g)\cdot h=(f\vee g)$. The dual of $U(X)$, $C(X)$, is the lattice of closed sets of X under inclusion and is thus determined by the algebraic structure of the idempotents of $L^+(X)$. If $L^+(X)$ has its idempotents isomorphic to those of $L^+(Y)$ then $C(X)$ is lattice isomorphic to $C(Y)$. Since both spaces are T_D spaces, by [3, Theorem 2.1], X is homeomorphic to Y. The topology of X can be explicitly recovered from $C(X)$ as in [1].

Received by the editors March 17, 1972.
AMS 1969 *subject classifications*. Primary 5420; Secondary 5460.
Key words and phrases. Lattice of closed sets, semicontinuous functions.

REFERENCES

1. D. Drake and W. J. Thron, *On the representations of an abstract lattice as the family of closed sets of a topological space*, Trans. Amer. Math. Soc. **120** (1965), 57–71. MR **32** #6390.

2. R. Neilsen and C. Sloyer, *Ideals of semi-continuous functions and compactifications of T_1 spaces*, Math. Ann. **187** (1970), 329–331. MR **42** #8450.

3. W. J. Thron, *Lattice-equivalence of topological spaces*, Duke Math. J. **29** (1962), 671–679. MR **26** #4307.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEBRASKA, LINCOLN, NEBRASKA 68508