

ON MIXING MEASURES FOR AXIOM A DIFFEOMORPHISMS

KARL SIGMUND

ABSTRACT. Let f be a diffeomorphism satisfying Smale's Axiom A and X an infinite basic set such that $f|X$ is topologically mixing. Let \mathcal{M} denote the space of f -invariant measures on X with the weak topology. It is shown that for a dense set of measures μ in \mathcal{M} , the system (X, f, μ) is Bernoulli. It follows that "in general" the elements of \mathcal{M} are weakly mixing.

1. Introduction. In [4], Bowen constructs an invariant measure for diffeomorphisms satisfying Smale's Axiom A which is in some sense the "best" measure, and in [5] he shows that under certain natural conditions the resulting dynamical system is a K -automorphism and, in fact, even measure theoretically isomorphic to a Bernoulli shift. In particular, it is strongly mixing and has positive entropy. In this note one considers the space of invariant measures provided with the weak topology and asks whether it contains many other measures with metric properties similar to those of Bowen's measure.

Bowen's theory of Markov partitions and results obtained in [14] allow one to approximate every invariant measure by Markov chains of positive entropy. If the diffeomorphism satisfies some mild condition, the approximating measures are strongly mixing and the corresponding dynamical system is measure theoretically isomorphic to a Bernoulli shift. As a corollary, one obtains that the invariant measures are generically weakly mixing.

In [7] and [16], theorems concerning invariant measures for Axiom A diffeomorphisms have been extended to the case of hyperbolic flows. One might expect that the results of this paper remain true for the space of invariant measures for C -dense Axiom A flows (see [7]).

2. Definitions and results. Let M be a compact C^∞ manifold without boundary and $f: M \rightarrow M$ a diffeomorphism. The nonwandering set Ω is the (closed, invariant) set of all $x \in M$ such that for every neighborhood U of x , there is an $n \neq 0$ with $T^n U \cap U \neq \emptyset$. f satisfies Axiom A if Ω is hyperbolic and the periodic points of f are dense in Ω (see [17, p. 777]). One

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can then write Ω as the finite disjoint union of basic sets Ω_s , where each Ω_s is closed invariant and $f|_{\Omega_s}$ is topologically transitive. By Bowen's C -density decomposition theorem [4, Theorem 2.7] one can further decompose each Ω_s into a finite disjoint union of closed sets $\Omega_{s,j}$ ($1 \leq j \leq m_s$) with $f(\Omega_{s,j}) = \Omega_{s,j+1}$ ($f(\Omega_{s,m_s}) = \Omega_{s,1}$) and $f^{m_s}|_{\Omega_{s,1}}$ topologically mixing. (Recall that a homeomorphism $g: X \rightarrow X$ is topologically mixing if for any pair of open sets $A, B \subset X$ there is an N such that $g^n(A) \cap B \neq \emptyset$ for all $n \geq N$.)

Every f -invariant (Borel) probability measure in \mathcal{M} has support in Ω and can be written as a finite convex combination of invariant measures concentrated on basic sets. Let Ω_s be a fixed basic set with infinitely many elements and let \mathcal{M} denote the space of f -invariant probability measures concentrated in Ω_s . \mathcal{M} is provided with the weak topology for measures (see [12, p. 40]).

Bowen constructs in [4] an element of \mathcal{M} which is nonatomic, positive on open sets, ergodic, and has positive entropy. (It is the unique measure maximizing entropy.) In [14] it was shown that there is a dense G_δ subset $\mathcal{N} \subset \mathcal{M}$ such that if $\mu \in \mathcal{N}$, then μ is nonatomic, positive on open sets, ergodic, and has entropy zero.

No $\mu \in \mathcal{M}$ can be weakly mixing if $f|_{\Omega_s}$ is not topologically mixing. But the restriction to the topologically mixing case is very natural: to every $\mu \in \mathcal{M}$ is associated an f^{m_s} -invariant measure $\bar{\mu}$ on $\Omega_{s,1}$ by $\bar{\mu}(E) = m_s \cdot \mu(E)$ for all Borel sets $E \subset \Omega_{s,1}$, and the correspondence $\mu \rightarrow \bar{\mu}$ is a homeomorphism. The system (Ω_s, f, μ_B) , where μ_B denotes Bowen measure, is a K -automorphism iff $f|_{\Omega_s}$ is topologically mixing (see [5, Theorem 34]). In this case, (Ω_s, f, μ_B) is measure theoretically isomorphic to a strongly mixing Markov chain and hence, by [9], to a Bernoulli shift. In [14] it was shown that strongly mixing measures form a set of first category in \mathcal{M} . The following theorem shows that they are dense.

THEOREM. *If $f|_{\Omega_s}$ is topologically mixing, then the measures μ such that (Ω_s, f, μ) is measure theoretically isomorphic to a Bernoulli shift form a dense subset of \mathcal{M} .*

COROLLARY 1. *If $f|_{\Omega_s}$ is topologically mixing then the measures which are mixing of order n are dense in \mathcal{M} (for all n). The set of weakly mixing measures is a dense G_δ in \mathcal{M} .*

COROLLARY 2. *The set of measures with positive entropy is dense in \mathcal{M} .*

Hyperbolic automorphisms of the n -torus are examples of Axiom A diffeomorphisms with only one basic set Ω_s and $f|_{\Omega_s}$ topologically mixing.

In [5], Bowen defines a continuous equivariant map π from a zero-dimensional dynamical system Σ onto Ω_s . The properties of π play a

central role in the proof of the theorem above. Bowen shows in [6] that $x \in \Sigma$ is periodic (or recurrent, or topologically transitive) iff $\pi(x)$ is periodic (or recurrent, or topologically transitive). One can look for further properties preserved under π . At the end of this paper there is a construction of a quasiregular $q \in \Omega_s$ such that $\pi^{-1}(q)$ contains a point which is not quasiregular.

3. Proofs. Let S denote the discrete space $\{1, \dots, S\}$ and $\theta(S)$ the space of all functions $x: \mathbb{Z} \rightarrow S$ from the integers \mathbb{Z} into S , provided with the product topology. Let σ denote the shift homeomorphism of $\theta(S)$:

$$[\sigma(x)]_i = x_{i+1} \quad \text{for } x \in \theta(S).$$

Let $T = (t_{ij})$ be an $S \times S$ -matrix of zeroes and ones. An element x of $\theta(S)$ is said to be T -admissible if $t_{x_i, x_{i+1}} = 1$ for all $i \in \mathbb{Z}$. One similarly defines T -admissible blocks $x_1 x_2 \cdots x_k$ of elements of S . The set Σ of all T -admissible $x \in \theta(S)$ is a compact metrizable space invariant under σ . (Σ, σ) is called the subshift of finite type associated with T .

In [5] Bowen proves that to every basic set Ω_s of the Axiom A diffeomorphism f there corresponds a subshift of finite type (Σ, σ) and a continuous surjective map $\pi: \Sigma \rightarrow \Omega_s$ such that the diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{\sigma} & \Sigma \\ \downarrow \pi & & \downarrow \pi \\ \Omega_s & \xrightarrow{f} & \Omega_s \end{array}$$

commutes.

LEMMA 1 [6, COROLLARY 11]. *If $x \in \Omega_s$ is periodic, then every $y \in \pi^{-1}(x)$ is periodic.*

LEMMA 2 [5, PROPOSITION 30]. *If $f|_{\Omega_s}$ is topologically mixing, then (Σ, σ) is also topologically mixing.*

It is easy to see that (Σ, σ) is topologically mixing iff T is irreducible and aperiodic (i.e. iff there is an N such that $t_{ij}^{(n)} > 0$ for all $n > N$ and all i, j).

If g is a homeomorphism of some space X and $x \in X$ a periodic point of period k , then the g -invariant measure μ_x which has mass $1/k$ at the points $x, gx, \dots, g^{k-1}x$ is called a periodic orbit measure (p.o.-measure).

LEMMA 3. *If (Σ, σ) is a subshift of finite type with periodic points dense and μ is an invariant measure, then μ can be approximated by p.o.-measures μ_x . If Σ has infinitely many elements, the period of x can be chosen arbitrarily large and prime.*

PROOF. By [3, Theorem 6.7], (Σ, σ) is topologically conjugate to a basic set of some Axiom A diffeomorphism of the 2-sphere. The lemma follows by [14, Theorem 1 and Lemma 1].

LEMMA 4. *If (Σ, σ) is a topologically mixing subshift of finite type with periodic points dense and $V(\mu)$ is a neighborhood of the invariant measure μ on Σ , then $V(\mu)$ contains a ρ such that the system (Σ, σ, ρ) is measure theoretically isomorphic to a Bernoulli shift.*

PROOF. By Lemma 3, there is a periodic point $x \in \Sigma$ of prime minimal period p such that $\mu_x \in V(\mu)$. Let S' denote the set of all T -admissible p -blocks of elements of S , arranged in such a way that the first p of the S' elements of S' are $(x_1 \cdots x_p), (x_2 \cdots x_p x_1), \dots, (x_p x_1 \cdots x_{p-1})$ (in this order). The transition rules given by T specify transition rules between the states of S' . Let $T' = (t'_{ij})$ denote the resulting $S' \times S'$ transition matrix and Σ' the corresponding subshift of $\theta(S')$. Since (Σ', σ) is topologically conjugate to (Σ, σ) , the matrix T' is irreducible and aperiodic. Let us write

$$T' = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix}$$

where A_0 is a $p \times p$ -submatrix. Since p is prime, A_0 must be of the form

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Indeed, if $(x_j x_{j+1} \cdots x_{j-1})$ is a successor of $(x_i x_{i+1} \cdots x_{i-1})$ according to the transition rules given by T' , it follows that $x_{i+1} = x_j$, and generally $x_m = x_n$ if $m - n = i + 1 - j$ (m, n modulo p). If $j \neq i + 1$, one would obtain $x_1 = x_2 = \cdots = x_p$, a contradiction to the choice of x . Since Σ is infinite, one has furthermore that A_0 is a proper submatrix of T' . Therefore, there exists a stochastic $S' \times S'$ matrix $L = (l_{ij})$ of the form

$$L = \begin{bmatrix} A_0 & 0 \\ C_1 & D_1 \end{bmatrix}$$

such that, for all (i, j) with $i > p$ or $j \leq p$, one has $l_{ij} > 0$ iff $t'_{ij} > 0$. For $n > 0$ one has

$$L^n = \begin{bmatrix} A_0^n & 0 \\ E(n) & D_1^n \end{bmatrix}.$$

The elements of $E(n)$ depend only on the l_{ij} with $i > p$ or $j \leq p$. Since these l_{ij} vanish iff the corresponding t'_{ij} vanish, and since T' is irreducible and aperiodic, there is an N such that all elements of $E(n)$ are positive for $n \geq N$.

The stochastic matrix L has at least one eigenvector $\lambda = (\lambda_1, \dots, \lambda_{S'})$ for the eigenvalue 1. Since λ is also an eigenvector of L^n , the vector $(\lambda_{p+1}, \dots, \lambda_{S'})$ is an eigenvector of D_1^n for the eigenvalue 1. But for $n \geq N$, the row sums of D_1^n (and hence its L^∞ -norm) are less than 1, and therefore $\lambda_j = 0$ for $p < j \leq S'$. Furthermore, the form of A_0 implies that $\lambda_1 = \lambda_2 = \dots = \lambda_p$. Hence the matrix L , although not irreducible, has only one normed eigenvector for the eigenvalue 1, given by $\bar{\lambda} = (1/p, \dots, 1/p, 0, \dots, 0)$. Remark that the invariant measure for (Σ', σ) given by the Markov shift induced by the transition matrix L and the initial probabilities $\bar{\lambda}$ (see [2, p. 30]) is just the measure μ_x .

Consider now the stochastic $S' \times S'$ -matrix P of the form

$$P = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$$

where A_1 is a $p \times p$ -matrix given by

$$\begin{bmatrix} 0 & 1 - \varepsilon_1 & 0 & \cdots & 0 \\ 0 & 0 & 1 - \varepsilon_2 & \cdots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 1 - \varepsilon_p & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and the elements of B_1 are positive iff the corresponding elements of the submatrix B_0 of T' are positive. Like T' , P is irreducible; hence, there exists a unique eigenvector $\bar{\pi} = (\pi_1, \dots, \pi_{S'})$ for the eigenvalue 1 (see, for example, Feller [8, p. 393]). If one chooses $\varepsilon_1, \dots, \varepsilon_p$ small enough, P approximates L and $\bar{\pi}$ approximates the unique normed eigenvector $\bar{\lambda}$ corresponding to the eigenvalue 1 of L . Thus the invariant measure ρ for (Σ', σ) given by the Markov shift induced by P and $\bar{\pi}$ can be chosen arbitrarily near the measure μ_x and therefore within $V(\mu)$. Since P is

irreducible and aperiodic, the system (Σ', σ, ρ) is a mixing Markov shift and hence (see [9]) measure theoretically isomorphic to a Bernoulli shift. Hence Lemma 4 is proved.

PROOF OF THE THEOREM. Let $\mu \in \mathcal{M}$ and a neighborhood $V(\mu)$ of μ be given. By [14, Theorem 1] there is a p.o.-measure μ_x in $V(\mu)$. Let y be an element of $\pi^{-1}(x)$. By Lemma 1, y is periodic. By Lemma 4, there is a sequence of measures ρ_n converging to the p.o.-measure μ_y such that all systems (Σ, σ, ρ_n) are (isomorphic to) Bernoulli shifts. The measures $\pi(\rho_n)$ converge to $\pi(\mu_y) = \mu_x$ and the systems $(\Omega_s, f, \pi(\rho_n))$ are—as homomorphic images of Bernoulli shifts—also Bernoulli shifts (see [10]). Hence $V(\mu)$ contains a $\rho \in \mathcal{M}$ such that (Ω_s, f, ρ) is measure theoretically isomorphic to a Bernoulli shift.

PROOF OF COROLLARY 1. The first part follows immediately from [13, p. 44]. The second follows from the proof of Theorem 2 in [11], which shows that if φ is a homeomorphism of a compact metric space X , the set of weakly mixing measures is a G_δ in the set of φ -invariant measures on X .

PROOF OF COROLLARY 2. If (Ω_s, f, ρ) is a Bernoulli shift, the entropy $h_\rho(f)$ of f with respect to ρ is positive. If $f|_{\Omega_s}$ is not topologically mixing, the corollary follows from Bowen's C -density decomposition theorem and the fact that $h_\mu(f^k) = k \cdot h_\mu(f)$ for $k > 0$ and μ f -invariant.

4. On quasiregular points. Let X be a compact metric space and φ a homeomorphism $X \rightarrow X$. A point $q \in X$ is called quasiregular if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} g(\varphi^i q)$$

exists for all continuous real-valued g on X . The map

$$g \rightarrow \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} g(\varphi^i q)$$

is a φ -invariant measure on X . In [15] it is shown that every $\mu \in \mathcal{M}$ is generated in this way by some quasiregular $q \in \Omega_s$.

One sees easily that if $q \in \Sigma$ is quasiregular, then $\pi(q)$ is quasiregular. We construct a $q \in \Sigma$ which is not quasiregular, but such that $\pi(q)$ is quasiregular.

Let (Ω_s, f) —and hence also (Σ, σ) —be topologically mixing. Let x denote a periodic point in Ω_s such that $\pi^{-1}(x)$ contains two periodic points $x^{(1)}$ and $x^{(2)}$ belonging to two different orbits.

(Such a situation occurs, for example, in [1, p. 30ff.], where (Ω_s, f) is the hyperbolic toral automorphism given by the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$. In [1] it is shown that the associated subshift of finite type is given by

the transition matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Let x be the fixed point of the torus, $x^{(1)}$ the sequence $(\cdots, 1, 1, 1, \cdots)$ and $x^{(2)}$ the sequence $(\cdots, 5, 5, 5, \cdots)$.

One can assume that $x^{(1)}$ and $x^{(2)}$ have the same (not necessarily minimal) period p . Let $A^{(j)}$ denote the T -admissible block $x_1^{(j)} \cdots x_p^{(j)}$ ($j=1, 2$). Let B_{12} denote a block of elements of S such that the block $A^{(1)}B_{12}A^{(2)}$ is T -admissible. (Such a B_{12} exists since (Σ, σ) is topologically mixing.) Similarly, let B_{21} be a block of elements of S such that $A^{(2)}B_{21}A^{(1)}$ is T -admissible. Let $m = \max(\text{card } B_{12}, \text{card } B_{21})$.

Let $q \in \theta(S)$ be given by

$$\begin{aligned} & \cdots A^{(1)} A^{(1)} B_{12} \cdot A^{(2)} A^{(2)} \cdots A^{(2)} B_{21} A^{(1)} A^{(1)} \cdots A^{(1)} \\ & \qquad (k_1 \text{ copies}) \qquad (k_2 \text{ copies}) \\ & \qquad \qquad \qquad \times B_{12} A^{(2)} A^{(2)} \cdots A^{(2)} B_{21} \cdots \\ & \qquad \qquad \qquad (k_3 \text{ copies}) \end{aligned}$$

(the dot denotes the decimal point between q_0 and q_1). Let k_j be an increasing sequence of integers such that

$$\frac{(k_1 + k_2 + \cdots + k_j)p + mj}{(k_1 + k_2 + \cdots + k_{j+1})p} < 2^{-j},$$

q is T -admissible, and hence in Σ . The iterates of q under σ follow the orbit of $x^{(2)}$ for a time, then the orbit of $x^{(1)}$ for a much longer time, then the orbit of $x^{(2)}$ for a still much longer time, etc. One can easily check that q is not quasiregular. But since π is continuous, and maps both $x^{(1)}$ and $x^{(2)}$ into x , the distance between $f^n(x)$ and $f^n(\pi(q))$ converges to 0, if one restricts n to the complement, in \mathbb{Z}^+ , of some set of density zero. It follows that $\pi(q)$ is quasiregular and generates the p.o.-measure μ_x .

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MATHEMATISCHES INSTITUT DER UNIVERSITÄT, A-1090 WIEN, STRUDLHOFGASSE 4,
VIENNA, AUSTRIA