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A NEW CHARACTERIZATION OF SEPARABLE GCR-ALGEBRAS

TROND DIGERNES

ABSTRACT. It is shown that a separable C^*-algebra $\mathfrak{A}$ is GCR if and only if the set of central projections in its enveloping von Neumann algebra $\mathfrak{B}$ is generated, as a complete Boolean algebra, by the set of open, central projections in $\mathfrak{B}$.

1. Let $\mathfrak{A}$ be a C^*-algebra, and $\mathfrak{B}$ its enveloping von Neumann algebra, that is, $\mathfrak{B}=\pi_u(\mathfrak{A})''$, where π_u is the direct sum of all cyclic representations of $\mathfrak{A}$. The representation π_u is faithful, and we may therefore consider $\mathfrak{A}$ as a sub-C^*-algebra of $\mathfrak{B}$. To each (nondegenerate) representation π of $\mathfrak{A}$ there corresponds a projection $E' \in \mathfrak{B}'=\pi_u(\mathfrak{A})'$ such that π may be identified with the map $A \in \mathfrak{A} \rightarrow AE' \in \mathfrak{B}E'$ [**2**, §§5 and 12]. A projection $E \in \mathfrak{B}$ is said to be open if it supports a left ideal in $\mathfrak{A}$; that is, if there is a left ideal J in $\mathfrak{A}$ such that $\bar{J}=\mathfrak{B}E$, where "–" denotes strong closure [**1**]. We let $\mathscr{P}$ denote the set of all central projections in $\mathfrak{B}$, $\mathscr{P}_0$ the set of open projections in $\mathscr{P}$ and $\langle\mathscr{P}_0\rangle$ the Boolean algebra generated by $\mathscr{P}_0$ in $\mathscr{P}$. With these notations the following has been proved by H. Halpern and the author [**5**]:

1. $\mathfrak{A}$ is CCR if and only if $\mathscr{P}_0$ is strongly dense in $\mathscr{P}$.
2. If $\mathfrak{A}$ is GCR, then $\langle\mathscr{P}_0\rangle$ is strongly dense in $\mathscr{P}$.

The purpose of this paper is to obtain a converse to 2, at least in the separable case.

For the general theory of C^*-algebras and von Neumann algebras we refer the reader to the two books of Dixmier ([**2**], [**3**]), especially §§4, 5 and 12 of [**2**].

2. With notations as above we have:

THEOREM. *For a separable C^*-algebra $\mathfrak{A}$ the following two conditions are equivalent:*

(i) $\mathfrak{A}$ *is* GCR;

(ii) $\langle\mathscr{P}_0\rangle$ *is strongly dense in* $\mathscr{P}$.

PROOF. (i)$\Rightarrow$(ii). See [**5**].

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(ii)$\Rightarrow$(i). To prove this we use the following characterization of separable GCR algebras, due to Glimm: $\mathfrak{A}$ is GCR if and only if any two irreducible representations of $\mathfrak{A}$ with the same kernel are equivalent [**4**].

So let π_1, π_2 be irreducible representations of $\mathfrak{A}$ with $\ker \pi_1 = \ker \pi_2$, and let Q_1, Q_2 be the central supports of the minimal projections in $\mathfrak{B}' = \pi_u(\mathfrak{A})'$ corresponding to π_1 and π_2 respectively. (The central support C_E of a projection E in a von Neumann algebra $\mathfrak{B}$ is defined by $C_E = \inf\{P \in \mathscr{P};\, PE = E\}$.) Then Q_1 and Q_2 are minimal in $\mathscr{P}$. It suffices to show that $Q_1 = Q_2$. We argue by contradiction: Suppose $Q_1 \neq Q_2$; then $Q_1Q_2 = 0$, by minimality. Let $\mathscr{P}_c$ denote the set of closed, central projections, i.e. $\mathscr{P}_c = \{I - P;\, P \in \mathscr{P}_0\}$ and set $\mathscr{P}^* = \mathscr{P}_0 \cup \mathscr{P}_c$.

Claim. There is a $P \in \mathscr{P}^*$ such that $Q_1 \leqq P$ and $Q_2 \leqq I - P$.

Assume, for a moment, this has been proved, and, for definiteness, let P be open. Then there is an ideal J in $\mathfrak{A}$ such that $\bar{J} = \mathfrak{B}P$, and consequently there is an $A \in J$ with $AQ_1 \neq 0$, since $0 \neq Q_1 \leqq P$. On the other hand, $AQ_2 = AP \cdot Q_2(I-P) = AQ_2P(I-P) = 0$, contradicting our assumption that $\ker \pi_1 = \ker \pi_2$, and we are through.

So it remains only to prove the Claim. Again we argue by contradiction: Assume there are distinct, minimal projections Q_1 and Q_2 in $\mathscr{P}$ such that,

(*) for all $P \in \mathscr{P}^*$, $(I - P)Q_1 \neq 0$ or $PQ_2 \neq 0$.

Let $Q = Q_1 + Q_2$ and consider the set:

$$\mathscr{P}(Q) = \{P \in \mathscr{P};\, PQ = Q \quad \text{or} \quad PQ = 0\}.$$

By (*) and by minimality of Q_1 and Q_2, $\mathscr{P}^* \subseteq \mathscr{P}(Q)$; and by minimality of Q_1 and Q_2 again, $\mathscr{P}(Q)$ is closed under finite unions, finite intersections and complementation. It follows that $\langle \mathscr{P}_0 \rangle = \langle \mathscr{P}^* \rangle \subseteq \mathscr{P}(Q)$. Now, by assumption there is a net $\{P_\alpha\}$ from $\langle \mathscr{P}_0 \rangle$ such that $P_\alpha \to Q_1$ strongly, and, by minimality of Q_1, we may assume $P_\alpha \geqq Q_1$ for all α. But then, since $\langle \mathscr{P}_0 \rangle \subseteq \mathscr{P}(Q)$, also $P_\alpha \geqq Q_1 + Q_2$ for all α, and consequently $Q_1 = \lim P_\alpha \geqq Q_1 + Q_2$, contradiction.

This completes the proof of the theorem.

3. Remark. In the course of the proof we have also established the following: If $\mathfrak{A}$ is a C^*-algebra (separable or not) with the property that $\langle \mathscr{P}_0 \rangle$ is dense in $\mathscr{P}$, then any two factor-representations of $\mathfrak{A}$ with the same kernel are quasi-equivalent.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024