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THE SUBCONTINUA OF A DENDRON FORM A HILBERT CUBE FACTOR[1]

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ABSTRACT. The title statement is proved, and it is shown further that the subcontinua of a dendron actually form a Hilbert cube when (and only when) the branch points of the dendron are dense. Along the way, it is established that whenever a Hilbert cube manifold is compactified into a Hilbert cube factor by the addition of another Hilbert cube factor having Property Z in the compactification, then the resulting space is actually a Hilbert cube.

It is known ([**1**], [**14**]) that the Cartesian product of any dendron with the Hilbert cube is homeomorphic to the Hilbert cube. In this paper, the hyperspace $C(D)$ of all nonvoid subcontinua of a dendron D is investigated and found to have the same property. In fact, it is shown that if the branch points of D are dense, then $C(D)$ is a Hilbert cube.

The method used presents in simple form the techniques employed elsewhere by the author and R. M. Schori ([**11**], [**12**]) to show that the hyperspace of all nonvoid, closed subsets of the unit interval is a Hilbert cube, and Theorem 1 is applied crucially in the extension of that result to finite, connected graphs in [**13**].

The space X will be termed a *Hilbert cube factor* if for some space Y, $X\times Y$ is homeomorphic to the Hilbert cube, Q. It is easy to see that this condition holds if and only if $X\times Q$ is a Hilbert cube. (See [**15**, Lemma 2].)

A *dendron* D is a *uniquely arcwise connected Peano continuum*. The *branch points* of D are those points which separate it into more than two

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components. Furthermore, each dendron D may be written as the closure in the plane of a countable union of arcs α_i, of lengths converging to zero, such that

$$\alpha_i \cap \left(T_{i-1} = \bigcup_{j=1}^{i-1} \alpha_j\right) = a_i,$$

is an endpoint of α_i [**16**]. This, together with the unique arcwise connectivity, allows the dendron to be written as the inv lim$\{T_i, r_i\}$, where $r_i: T_i \to T_{i-1}$ is defined by retracting α_i to a_i. (The r_i's extend to retractions of D.) Finally, a dendron always admits a *convex* metric ρ, i.e., one for which there always exists a point halfway between any two given points. The metric ρ may always be chosen so that b separates a from c in D if and only if the points a, b, and c are distinct and $\rho(a, c) = \rho(a, b) + \rho(b, c)$. (See [**14**] for a simple construction of such a metric by embedding in the Hilbert cube.)

We are concerned with the hyperspace $C(D)$ of all (nonvoid) subcontinua of D under the topology generated by the Hausdorff metric d_H, which is defined by the condition that $d_H(K_1, K_2) < \varepsilon$ if and only if each point of K_1 is within ε of some point of K_2 and *vice versa*. The inverse limit expression inv lim$\{T_i, r_i\}$ of D given above induces one inv lim$\{C(T_i), \bar{r}_i\}$ of $C(D)$ when $\bar{r}_i$ is the map $C(T_i) \to C(T_{i-1})$ induced by r_i. We shall show that this expression guarantees that $C(D)$ is a Hilbert cube factor, but first it is necessary to introduce some notation.

A closed subset A of a separable, metric ANR X has *Property* Z *in* X if and only if for every open set U of X the inclusion map $U \backslash A \to U$ is a homotopy equivalence. (This is the appropriate generalization to ANR's of R. D. Anderson's definition [**2**] of Property Z in infinite-dimensional manifolds. See, for example, the article by Eells and Kuiper [**9**], in which it is shown that A has Property Z in X if for every $a \in A$ there are arbitrarily small open neighborhoods U of a in X with the property that each map of a sphere into U may be deformed in U to a map into $U \backslash A$, and each map $f: (B^n, S^{n-1}) \to (U, U \backslash A)$ is homotopic in U to a map $g: B^n \to U \backslash A$ by a homotopy which is constant on $S^{n-1} \times I$. See also [**4**]. This condition is guaranteed whenever there is a homotopy $H: X \times I \to X$ with $H(x, 0) = x$ for each $x \in X$ which has the property that $H(X \times (0, 1]) \cap A = \varnothing$.) The importance of Property Z in the Hilbert cube is expressed by Anderson's homogeneity theorem [**2**, Theorem 10.1]: *Each homeomorphism between two subsets of the Hilbert cube with Property* Z *extends to a homeomorphism of the Hilbert cube.* This theorem will be used immediately to establish Theorem 1 below, which gives a sufficient condition for the compactification of a Hilbert cube manifold to be a Hilbert cube.

THEOREM 1. *Let X be a compactification of a Hilbert cube manifold M such that both X and the remainder, $A=X\backslash M$, are Hilbert cube factors. If A has Property Z in X, then X is a Hilbert cube.*

PROOF. The pair (X, A) is a pair of Hilbert cube factors, so $(X\times Q, A\times Q)$ is a pair of Hilbert cubes. Moreover, $A\times Q$ has Property Z in $X\times Q$, as is easily seen. The homogeneity theorem then allows a homeomorphism of pairs $(X\times Q, A\times Q)\to(A\times Q\times I, A\times Q\times\{1\})$ extending the natural homeomorphism on the second entries. Thus, the space Y obtained from $X\times Q$ by simultaneously identifying $\{a\}\times Q$ to a point a for each a in A is homeomorphic to the result, Z, of identifying each $\{a\}\times Q\times\{1\}$ to a point in $A\times Q\times I$. However, Z is just the product of A with the cone $C(Q)$ of Q. Note that $C(Q)$ is a Hilbert cube: One way of seeing this is to observe that Q, hence $C(Q)$, may be embedded in Hilbert space as a compact, convex, infinite-dimensional set, so the theorem of O. H. Keller, **[10]** that all such are Hilbert cubes applies. Since $C(Q)$ is a Hilbert cube and A is a Hilbert cube factor, Z is also a Hilbert cube, and so is Y. There remains only to show that Y is homeomorphic to X, which is done as follows: $Y\backslash A$ is homeomorphic to $X\backslash A$ by [3], because it is homeomorphic to $X\times Q\backslash A\times Q=M\times Q$ and each Hilbert cube manifold is homeomorphic to its product with a Hilbert cube. Also, Y/A, the result of identifying A to a point, is a Hilbert cube because Y/A is homeomorphic to $Z/A=C(A\times Q)$ which is homeomorphic to a Hilbert cube. Hence, $Y\backslash A$, and therefore M, embeds in Q as an open subset. Now, the proof of the theorem in the addendum of [3] shows that for any open cover $\mathcal{U}$ of M there is a homeomorphism $h: M\times Q\to M$ which is a $\mathcal{U}$-close to the projection $p: M\times Q\to M$, i.e., the cover $\{\{p(x), h(x)\}|x\in M\times Q\}$ refines $\mathcal{U}$. The proper choice of $\mathcal{U}$ yields a homeomorphism of $Y\backslash A$ onto M extending to a homeomorphism of Y onto X which is the identity on A. Since Y is a Hilbert cube, so is X.

The following is the main theorem of this note.

THEOREM 2. *The subcontinua $C(D)$ of a dendron D form a Hilbert cube factor which is a Hilbert cube if (and only if) the branch points of D are dense.*

PROOF. Write $D=\operatorname{inv}\lim\{T_i, r_i\}$ and for each $i>1$ let b_i be the endpoint of the arc $\alpha_i=\operatorname{cl}(T_i\backslash T_{i-1})$ which does not lie in T_{i-1}. Now consider

$$r_i' = \bar{r}_i \mid C_{b_i}(T_i): C_{b_i}(T_i) \to C(T_{i-1}),$$

where $C_{b_i}(T_i)$ denotes those members of $C(T_i)$ containing b_i. Now let $M_{r_i'}$ be the mapping cylinder of r_i', that is,

$$M_{r_i'} = [C_{b_i}(T_i) \times I \cup C(T_{i-1})]/\Xi,$$

where Ξ is the upper-semicontinuous decomposition with nondegenerate elements all sets of the form $\{K\} \cup (r_i'^{-1}(K) \times \{0\})$ where K is in $C_{a_i}(T_{i-1})$.

There is a homeomorphism g_i of $C(T_i)$ onto $M_{r_i'}$ for which the following diagram commutes, c_i being the collapse of $M_{r_i'}$ to $C(T_{i-1})$ given by $[K, t] \to r_i'(K)$ if K is in $C_{b_i}(T_i)$ and $[K] \to K$ if K is in $C(T_{i-1})$.

$$\begin{tikzcd} C(T_i) \arrow[rr, "g_i"] \arrow[dr, "\bar{r}_i"'] & & M_{r_i'} \arrow[dl, "c_i"] \\ & C(T_{i-1}) & \end{tikzcd}$$

This g_i may be defined by parameterizing α_i with $[0, 1]$ so that $a_i \leftrightarrow 0$ and $b_i \leftrightarrow 1$ and then letting

$$\begin{aligned} g_i(K) &= [K], && \text{if } K \cap \alpha_i = \varnothing, \\ &= [K \cup \alpha_i, d], && \text{if } a_i \in K, \\ &= [(1/d)K, d], && \text{if } K \subset \alpha_i \backslash \{a_i\}, \end{aligned}$$

where $d = \sup(K \cap \alpha_i)$ if $K \cap \alpha_i \neq \varnothing$.

Both $C(T_i)$ and $C_{b_i}(T_i)$ are polyhedra by **[6]** and **[8]** (see also [7] for a definitive analysis), and they are easily seen to be contractible because T_i is contractible to b_i. Therefore, by **[14]** they are Hilbert cube factors, so by **[15]**,

$$c_i \times \mathrm{id}: M_{r_i'} \times Q \to C(T_{i-1}) \times Q$$

is a uniform limit of homeomorphisms. The commutative diagram above shows that

$$\bar{r}_i \times \mathrm{id}: C(T_i) \times Q \to C(T_{i-1}) \times Q$$

is also a uniform limit of homeomorphisms. Thus,

$$\operatorname{inv} \lim\{C(T_i) \times Q, \bar{r}_i \times \mathrm{id}\},$$

being an inverse limit of Hilbert cubes and uniform limits of homeomorphisms, is a Hilbert cube by [5]. However, $\operatorname{inv}\lim\{C(T_i) \times Q, \bar{r}_i \times \mathrm{id}\}$ is easily seen to be homeomorphic to $\operatorname{inv}\lim\{C(T_i), \bar{r}_i\} \times Q = C(D) \times Q$, so $C(D)$ is a Hilbert cube factor.

If now, the branch points of D are *not* dense, it is immediate that $C(D)$ contains an open 2-cell, namely, those nondegenerate members which lie entirely within some open arc which is an open subset of D. Therefore, in this case $C(D)$ is not itself a Hilbert cube.

On the other hand, if the branch points of D are dense, Theorem 1 may be employed to show $C(D)$ a Hilbert cube by identifying D with the set of all degenerate subcontinua of itself and showing that (1) it has Property Z in $C(D)$, and (2) $C(D)\backslash D$ is a Hilbert cube manifold. (D itself is a Hilbert cube factor as may be seen by using the proof already given for $C(D)$ or by citing **[14]**. The fact was originally proven by R. D. Anderson **[1]**, but his proof was never published.)

It is easy to show that D has Property Z in $C(D)$, because, using the metric ρ selected at the outset, the homotopy $H: C(D)\times I\to C(D)$ sending (K, t) to the closed t-neighborhood of K in D satisfies the homotopy condition mentioned parenthetically which guarantees Property Z. There remains, then, only to verify that $C(D)\backslash D$ is a Hilbert cube manifold. Again using the metric ρ, it is easy to see that for any nondegenerate subcontinuum K of D there are an $\varepsilon>0$ and two other nondegenerate subcontinua K_- and K_+ of D with the property that the closed ε-neighborhood N of K in $C(D)$ is the set of all members of $C(D)$ containing K_- yet lying in K_+. Setting $H=K_+/K_-$, it is easy to see that the map $C(K_+)\to C(H)$ induced from $K_+\to K_+/K_-$ carries N homeomorphically to $C_*(H)$, where $*=K_-/K_-$. Also, because the branch points of D are dense, the branch points of K_+ lying in K_- are dense in K_-, and thus $*$ separates H into infinitely many components with closures H_i, $i=1, 2, \cdots$. Now, the intersection maps $\bigcap H_i: C_*(H)\to C_*(H_i)$ are continuous, and their product $\bigcap: C_*(H)\to \prod_{i=1}^{\infty} C_*(H_i)$ is a homeomorphism. Furthermore, the proof that $C(D)$ is a Hilbert cube factor will also apply without modification to show that each $C_*(H_i)$ is one also. Therefore, N is homeomorphic to a countably infinite product of nondegenerate Hilbert cube factors and by **[14]** is itself a Hilbert cube. This shows that $C(D)\backslash D$ is a Hilbert cube manifold, so Theorem 1 applies to show that $C(D)$ is a Hilbert cube.

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