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UNIVERSAL REGRESSIVE ISOLS

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ABSTRACT. E. Ellentuck introduced universal isols in Math. Z. **98** (1967), 1–8, to show how counterexamples in the arithmetic of the isols may be obtained in a uniform manner. Also Ellentuck was the first to prove, in unpublished notes, that there will be regressive isols that are universal. The present paper contains a relatively short proof that every infinite multiple-free regressive isol will be universal.

1. **Preliminaries and basic ideas.** We shall assume that the reader is familiar with the concepts and main results cited in the papers [**1**], [**2**] and [**4**]. We adopt the notation of [**2**]. The principal result proved in [**2**] is the following lemma.

LEMMA 1. *Let α be a nonempty recursive set and let f be any increasing recursive function whose range is α. Let α_R denote the set of regressive isols that belong to the extension of α to the isols, and let D_f denote the canonical extension of f to the isols. Then $\alpha_R = D_f(\Lambda_R)$.*

The basic idea for the proof of the main theorem in this paper arises from observing an elementary yet useful way of characterizing α and α_R for particular recursive sets α. This way is given in the following lemma and theorem. The techniques employed in the proof of the theorem are similar to those in [**1**] and [**2**].

LEMMA 2. *Let α be an infinite recursive set of numbers and let f denote the principal function of α. Then there will be functions g and h such that*

(1) *g and h are each increasing and recursive,*

(2) *g ranges over an infinite set, and*

(3) *$f(x)=2\cdot g(x)+h(x)$, for each number x, if and only if the complement of α is also an infinite set.*

PROOF. Assume first that there are functions g and h having the properties (1)–(3). Let us also assume that the complement of α is a finite set. Then there would be a number k such that

$$f(k) = y,\ \ f(k+1) = y+1,\ \ f(k+2) = y+2, \cdots.$$

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Since both g and h are increasing functions, it would then follow from property (3), that

$$g(k+1) = g(k+2) = g(k+3) = \cdots .$$

Therefore g would have a finite range, and this would contradict property (2). Therefore it must be that the complement of α is infinite.

Let us assume now that both α and its complement are infinite sets. Define functions g and h in the following way. Let $g(0)=0$ and let $h(0)=f(0)$. Assume that the values for g and h have been defined for values up to and including the number y. To define the functional values at $y+1$, we consider two cases. Let $a=f(y)$ and let $b=f(y+1)$.

Case 1. $b=a+1$. Define

$$g(y+1) = g(y), \quad \text{and} \quad h(y+1) = 1 + h(y).$$

Case 2. $b=a+u+2$ with $u \geqq 0$. Define

$$g(y+1) = 1 + g(y), \quad \text{and} \quad h(y+1) = u + h(y).$$

Because α is an infinite recursive set and f is its principal function, f will be a strictly increasing recursive function. Combining this property with the definitions above, it is easy to see that properties (1) and (3) will be true. Also, because the complement of α is an infinite set, Case 2 above will occur infinitely often, and this will mean that g will range over an infinite set. This gives property (2), and also proves the lemma.

THEOREM. *Let α be an infinite recursive set and let Y be an infinite regressive isol belonging to α_R. If the complement of α is also an infinite set, then there will be regressive isols S and T with S infinite and $Y=2S+T$.*

PROOF. Let f denote the recursive principal function of α. Assume that the complement of α is an infinite set and let g and h be functions chosen to have the properties as in Lemma 2. By Lemma 1 we know that $\alpha_R = D_f(\Lambda_R)$. Also, by combining property (3) in Lemma 2 with the well-known metatheorem of A. Nerode for such statements, it follows that $D_f(C) = 2D_g(C) + D_h(C)$, for all isols C. Because Y is in α_R then $Y = D_f(U)$ for some regressive isol U. Therefore, also $Y = D_f(U) = 2D_g(U) + D_h(U)$.

Since Y is infinite U will be also, and because g and h are each increasing and recursive functions $D_g(U)$ and $D_h(U)$ will each be regressive isols. Finally, we would like to note that $D_g(U)$ will be an infinite isol. This property may be verified in the following way. Consider the value of $D_g(U)$ expressed as an infinite series of isols, as given in [**1**, Proposition 2], and observe that in this series the associated e-difference function of g will be positive infinitely often since the range of g is an infinite set. Combining this form of an infinite series representation of $D_g(U)$ with the property that

U is an infinite regressive isol, it is easy to verify that the value of $D_g(U)$ will be infinite. If we set $S = D_g(U)$ and $T = D_h(U)$ then the desired result of the theorem will follow.

2. **Universal and multiple-free isols.** In view of [**5**, p. 4], a universal regressive isol may be defined in the following way: A regressive isol U is *universal* if, for every recursive set α, $U \in \alpha_R$ implies the complement of α is a finite set. An infinite isol Y is called *multiple-free*, if for every isol B, $2B \leqq Y$ implies B is a finite isol. Multiple-free isols were introduced and studied in [**4**]. An example of an infinite regressive isol that is multiple-free appears in [**3**]. We can obtain directly from the theorem in §1 the following result, and it is the main theorem of the paper.

THEOREM. *Every infinite multiple-free regressive isol is universal.*

M. Hassett has shown that there will be universal regressive isols that are not multiple-free (not yet published). From this result we see that the converse of the previous theorem will not be true.

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