A ROLE FOR DOUBLY STOCHASTIC MATRICES IN GRAPH THEORY

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ABSTRACT. This paper represents a strongly connected digraph as a doubly stochastic matrix. It then uses this doubly stochastic representation to prove several theorems concerning the critical arcs of strongly connected graphs.

Definitions and notation. The definitions concerning matrix theory for the most part may be found in [3]. All others will be given below.

An $n \times n$ nonnegative matrix $A = (a_{ij})$, so that $\sum_R a_{iR} = \sum_R a_{Ri} = \lambda$ for each $i \in \{1, 2, \cdots, n\}$, is called a λ-doubly stochastic matrix. If $\lambda = 1$, the matrix is simply called doubly stochastic. A pair of matrices $A = (a_{ij})$ and $B = (b_{ij})$ such that $a_{ij} = 0$ if and only if $b_{ij} = 0$ are said to have the same 0-pattern. If there is a doubly stochastic matrix B which has the same 0-pattern as A, then A is said to have doubly stochastic pattern. We say that A has a loop if A has distinct positive entries

$$a_{i_1 j_1}, a_{i_1 j_2}, a_{i_2 j_2}, a_{i_2 j_3}, \cdots, a_{i_r j_r}, a_{i_r j_{r+1}}, a_{i_{r+1} j_{r+1}} = a_{i_1 j_1}.$$

Finally, let E_{ij} denote the $n \times n$ (0, 1)-matrix with a one only in the ijth position.

All definitions concerning graph theory can be found in [1] and [5].

Results. We first establish a relation between doubly stochastic matrices and digraphs. For this we include the following two theorems.

THEOREM 1. *A digraph G is strongly connected if and only if its associated matrix A is irreducible* [**10**, p. 20].

THEOREM 2. *If A is an irreducible matrix, then there is a diagonal matrix D with positive main diagonal so that DAD^{-1} has its ith row sum equal to its ith column sum for each $i \in \{1, 2, \cdots, n\}$* [**6**, p. 3].

Now a positive main diagonal may be added to the matrix DAD^{-1} to yield a λ-doubly stochastic matrix where $\lambda > 1$. Hence the relation between doubly stochastic matrices and digraphs may be stated as:

THEOREM 3. *A digraph is strongly connected if and only if there is a doubly stochastic matrix M with positive main diagonal so that*
(a) *M is irreducible, and*
(b) $M = (m_{ij})$ *is such that if $i \neq j$ then $m_{ij} > 0$ if and only if there is an arc from x_i to x_j.*

Any such matrix M is called a doubly stochastic model of G. We now give three lemmas concerning doubly stochastic matrices.

LEMMA 1. *Suppose A is nonnegative with a positive main diagonal. Then the following two statements are equivalent.*
(a) *A has doubly stochastic pattern.*
(b) *There is a permutation matrix P so that PAP^T is a direct sum of irreducible matrices, i.e. A is completely reducible.*

PROOF. We first show that (a) implies (b). For this we may assume that A is doubly stochastic. If A is irreducible we are through. If A is reducible, then there is a permutation matrix P so that $PAP^T = \begin{pmatrix} X & 0 \\ Z & Y \end{pmatrix}$ where X and Y are square. Suppose X is $k \times k$. Then $\sum_{i,j} x_{ij} = k$ by considering the first k row sums of PAP^T. Considering the first k column sums of PAP^T we have $\sum_{i,j} x_{ij} + \sum_{i,j} z_{ij} = k$. Hence $\sum_{i,j} z_{ij} = 0$ and $PAT^T = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$. Since X and Y are now doubly stochastic the theorem follows by induction.
That (b) implies (a) is an application of Theorem 2.

LEMMA 2. *If A is an irreducible doubly stochastic matrix with a positive main diagonal and $i \neq j$, then the following two statements are equivalent.*
(a) $A - a_{ij}E_{ij}$ *has doubly stochastic pattern.*
(b) $A - a_{ij}E_{ij}$ *is irreducible.*

PROOF. We first show that (a) implies (b). If $A - a_{ij}E_{ij}$ has doubly stochastic pattern then by Lemma 1 it is irreducible or completely reducible. If $A - a_{ij}E_{ij}$ is completely reducible, then A is reducible. Hence $A - a_{ij}E_{ij}$ must be irreducible.
That (b) implies (a) is a consequence of Theorem 2.

LEMMA 3. *If A is a doubly stochastic matrix with positive main diagonal then there is a doubly stochastic matrix B with the same 0-pattern as A so that each main diagonal entry of B is larger than each nondiagonal entry of B.*

PROOF. Suppose λ is some number so that $0 < \lambda < 1$. Then $\lambda A + (1 - \lambda)I$ is doubly stochastic with the same 0-pattern as A. Hence, for λ sufficiently small, Lemma 3 follows.

We are now in a position to show how doubly stochastic matrices may be used in the analysis of critical arcs in strongly connected digraphs.

THEOREM 4.   *The arc x_{ij} from vertex v_i to vertex v_j of a strongly connected digraph G is critical if and only if there is not a doubly stochastic model M of G with m_{ij} smaller than all other nonzero entries in M.*

PROOF.   Suppose x_{ij} is not critical. Then $G - x_{ij}$ is still strongly connected. Hence there is a doubly stochastic model $N = (n_{ij})$ of $G - x_{ij}$. x_{ij} is on an elementary circuit in G, say $x_{ij}, x_{jj_1}, x_{j_1 j_2}, \cdots, x_{j_R i}$. Suppose $\varepsilon > 0$. Add ε to $n_{ij}, n_{jj_1}, n_{j_1 j_2}, \cdots, n_{j_R i}$ in N and subtract ε from n_{ii}, n_{jj}, $n_{j_1 j_1}, \cdots, n_{j_R j_R}$ in N yielding the matrix $N' = (n'_{ij})$. For ε sufficiently small, N' is a doubly stochastic model of G and n_{ij} is smaller than all other nonzero entries in M.

Suppose x_{ij} is critical. Further suppose G has a doubly stochastic model M so that m_{ij} is smaller than all other nonzero entries in M. Subtract m_{ij} from $m_{ij}, m_{jj_1}, \cdots, m_{j_R i}$ in M and add m_{ij} to $m_{ii}, m_{jj}, m_{j_1 j_1}, \cdots$, $m_{j_R j_R}$ in M yielding the matrix M'. M' is a doubly stochastic model of $G - x_{ij}$. Lemma 2 then implies that M' is irreducible. Hence $G - x_{ij}$ is strongly connected. This contradicts the hypothesis that x_{ij} is critical.

THEOREM 5.   *If G is a strongly connected digraph, then given any elementary circuit C of G there is a nonempty set of arcs from C whose removal from G leaves*
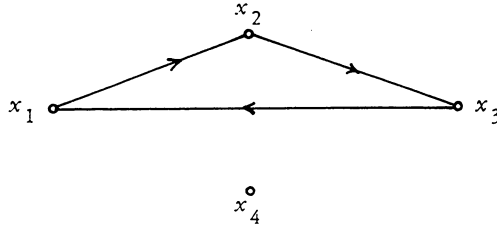
(a) *a strongly connected digraph, or*

(b) *a disjoint union of strongly connected digraphs.*

PROOF.   Suppose the elementary circuit C contains the following arcs: $x_{i_1 i_2}, x_{i_2 i_3}, \cdots, x_{i_R i_1}$. Suppose M is a doubly stochastic model of G. By Lemma 3 we may assume every entry m_{RR} of M is larger than every entry m_{ij} $(i \neq j)$ of M. Now consider $m_{i_1 i_2}, m_{i_2 i_3}, \cdots, m_{i_R i_1}$ in M. Let $m = \min[m_{i_1 i_2}, \cdots, m_{i_R i_1}]$. Now subtract m from each of $m_{i_1 i_2}, m_{i_2 i_3}, \cdots, m_{i_R i_1}$ in M and add m to $m_{i_1 i_1}, m_{i_2 i_2}, \cdots, m_{i_R i_R}$ in M. This results in a doubly stochastic matrix M'. The result now follows from Theorem 3 and Lemma 1.

Note that the theorem does not imply that any of the arcs of C may be arbitrarily chosen as can be seen by considering the example below:
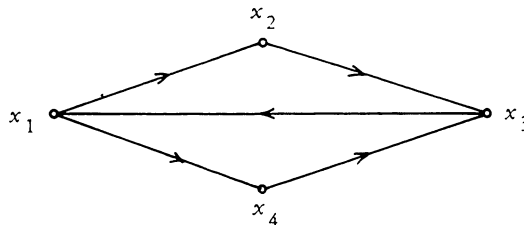
Let $C:x_{14}$, x_{43}, x_{31}. The arcs x_{14} and x_{43} may be removed to yield two disjoint strongly connected digraphs, i.e.

however, x_{31} may not be removed. An implication of this theorem is, of course, that a minimally strongly connected digraph is the connection through an elementary circuit of minimally strongly connected digraphs with fewer vertices.

The next theorem of this section is also in [4, p. 18]; however, we feel that our proof is the more elementary.

THEOREM 6.   *Suppose G is a minimally connected digraph with symmetric arcs x_{ij} and x_{ji}. If x_{ij} and x_{ji} are removed from G yielding G', then G' is a disjoint union of two strongly connected digraphs.*

PROOF.   Let M be a doubly stochastic model of G. Suppose ε is a positive number with $0 < \varepsilon < \min[m_{ij}, m_{ji}]$. Subtract ε from m_{ij} and m_{ji} in M and add ε to m_{ii} and m_{jj} in M. This yields a doubly stochastic matrix M' which is a model for G. Further for sufficiently large ε we see that m'_{ij} or m'_{ji} is a smallest positive entry in M'. Without loss of generality suppose $m'_{ij} \leq m'_{ji}$. Since x_{ij} is critical, $M' - m'_{ij}E_{ij}$ is reducible. Hence there is a permutation matrix P so that $P[M - m'_{ij}E_{ij}]P^T = \left(\begin{smallmatrix} X & 0 \\ Z & Y \end{smallmatrix}\right)$ where X and Y are square. Suppose X is $k \times k$. Then $\sum_{ij} x_{ij} = k - m'_{ij}$. Further $\sum_{ij} x_{ij} + \sum_{ij} z_{ij} = k$. Hence $\sum_{ij} z_{ij} = m'_{ij}$. Since m'_{ji} is in Z we see that Z has only one positive entry. Hence the removal of x_{ij} and x_{ji} from G results in two disjoint digraphs. It is easily seen that these two digraphs are strongly connected and hence the theorem is established.

We include the next two theorems to show some slightly different techniques of argument in the use of doubly stochastic matrices.

THEOREM 7.   *Suppose G is a strongly connected digraph with C_1 and C_2 elementary circuits of G. Suppose C_1 and C_2 have one and only one arc x in common. Then there is a nonempty set of arcs S of C_1 so that $x \notin S$ and*
(a) *$G - S$ is a strongly connected digraph, or*
(b) *$G - S$ is a disjoint union of strongly connected digraphs.*

PROOF. Suppose M is a doubly stochastic model of G so that M is $n \times n$ and $m_{ii} > m_{pq}$ for all $i, p, q \in \{1, 2, \cdots, n\}$ with $p \neq q$. Suppose $C_1 : x_{i_1 i_2}, x_{i_2 i_3}, x_{i_R i_1}; C_2 : x_{j_1 j_2}, x_{j_2 j_3}, \cdots, x_{j_S j_1}$. Let

$$m = \max\{m_{i_1 i_2}, m_{i_2 i_3}, \cdots, m_{i_R i_1}\}.$$

Now add m to $m_{j_1 j_2}, m_{j_2 j_3}, \cdots, m_{j_S j_1}$ in M and subtract m from $m_{j_1 j_1}$, $m_{j_2 j_2}, \cdots, m_{j_S j_S}$ obtaining a doubly stochastic model M' of G. M' has the property that $\max\{m'_{i_1 i_2}, m'_{i_2 i_3}, \cdots, m'_{i_R i_1}\} = m'_{ij}$ where $x_{ij} = x$. Further, m'_{ij} is larger than every other number in $\{m'_{i_1 i_2}, m'_{i_2 i_3}, \cdots, m'_{i_R i_1}\}$. The result now follows by an argument similar to that of Theorem 5.

THEOREM 8. *If G is a strongly connected digraph and C a closed semipath in G then there is a nonempty set of arcs S from C so that*
(a) *$G - S$ is a strongly connected digraph, or*
(b) *$G - S$ is a disjoint union of strongly connected digraphs.*

PROOF. Let M be a doubly stochastic model of G so that every main diagonal entry is larger than every off diagonal entry. Let $C : v_{i_1}, x_{i_1 j_1}, v_{i_2}, x_{i_2 j_2}, \cdots, v_{i_S}, x_{i_S j_S}, v_{i_1}$ where v_{i_R} each is a vertex of G and $x_{i_R j_R}$ an arc between v_{i_R} and $v_{i_{R+1}}$ $(k+1 \bmod S)$. Consider the following entries in M: $\{m_{i_R j_R}$ where $R \in \{1, 2, \cdots, S\}\} \cup \{m_{i_R i_R}$ if $x_{i_{R-1} j_{R-1}}$ is an arc into v_R and $x_{i_R j_R}$ an arc out of $v_R\} = N$. Now the entries of N in M form a loop of $M : m_{t_1 t_2}, m_{t_3 t_2}, M_{t_3 t_4}, \cdots, m_{t_r t_{r+1}}, m_{t_{r+2} t_{r+1}}, m_{t_{r+2} t_{r+3}} = m_{t_1 t_2}$. Again, without loss of generality we may assume $\min_{m_{ij} \in N}\{m_{ij}\} = m_{t_1 t_2}$. Subtract $m_{t_1 t_2}$ from each of $m_{t_1 t_2}, m_{t_3 t_4}, \cdots, m_{t_r t_{r+1}}$ in M and add $m_{t_1 t_2}$ to each of $m_{t_3 t_2}, \cdots, m_{t_{r+2} t_{r+1}}$ in M yielding the matrix M'. M' is now a doubly stochastic model of $G - S$ and the result follows immediately.

The above two theorems may well shed some additional light on the structure of minimally connected digraphs.

In conclusion we list [2], [7], and [9] as papers relevant to our topic. Brualdi, in [2], gives an extensive discussion of the kinds of row and column sums which can be obtained when the 0-pattern of a matrix is specified. Sinkhorn and Knopp in [9] give an interesting substructure of fully indecomposable matrices through the use of doubly stochastic matrices.

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