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MAXIMAL WEAK-* DIRICHLET ALGEBRAS[1]

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ABSTRACT. The purpose of this note is to demonstrate, in the context of weak-* Dirichlet algebras, the interdependence of a number of properties possessed by the space of bounded analytic functions in the open unit disc.

The space of bounded analytic functions in the open unit disc H^∞ has a number of properties, among which are these: (i) H^∞ is an integral domain; (ii) the boundary values of any nonzero function in H^∞ cannot vanish on a set of positive Lebesgue measure; and (iii) the space of boundary functions of functions in H^∞ forms a maximal weak-* closed subalgebra in L^∞ (of Lebesgue measure on the unit circle). Property (i) is quite elementary, while the other two are fairly deep facts and one would not expect much of a relation to hold between them. Surprisingly, however, from an axiomatic point of view, these three properties are equivalent. We shall show this in the context of weak-* Dirichlet algebras which were introduced by Srinivasan and Wang [**4**].

Recall that by definition a weak-* Dirichlet algebra is an algebra $\mathfrak{A}$ of essentially bounded measurable functions on a probability measure space $(X, \mathscr{S}, \sigma)$ such that (i) the constant functions lie in $\mathfrak{A}$; (ii) $\mathfrak{A}+\bar{\mathfrak{A}}$ is weak-* dense in $L^\infty(\sigma)$ (the bar denotes conjugation, here and always); and (iii) for all φ and ψ in $\mathfrak{A}$, $\int_X \varphi\psi\, d\sigma=(\int_X \varphi\, d\sigma)(\int_X \psi\, d\sigma)$. The abstract Hardy spaces $H^p(\sigma)$, $1\leq p\leq\infty$, associated with $\mathfrak{A}$ are defined as follows. For $1\leq p<\infty$, $H^p(\sigma)$ is the $L^p(\sigma)$-closure of $\mathfrak{A}$, while $H^\infty(\sigma)$ is defined to be the weak-* closure of $\mathfrak{A}$ in $L^\infty(\sigma)$.

Our goal in this note is the proof of the following

THEOREM. *The following properties which $H^\infty(\sigma)$ may possess are equivalent*:

(1) *$H^\infty(\sigma)$ is an integral domain*;

(2) *no nonzero function in $H^\infty(\sigma)$ can vanish on a set of positive measure*; *and*

(3) *$H^\infty(\sigma)$ is a maximal weak-* closed subalgebra of $L^\infty(\sigma)$.*

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This result was motivated by the considerations of Gamelin in [**1**, Chapter VII, §8] as well as by those of Merrill in [**3**]. Merrill pointed out that there are weak-* Dirichlet algebras such that $H^\infty(\sigma)$ fails to have any one of the properties listed in the theorem. Finally, observe that there are certain formal similarities between our theorem and some of the results of Hoffman and Singer in [**2**].

PROOF THAT (3) IMPLIES (2). For any measurable set E, let χ_E denote its characteristic function. Suppose φ is in $H^\infty(\sigma)$ and is supported on the set E, $0<\sigma(E)<1$. If $\mathscr{M}$ is the closure of $\varphi H^2(\sigma)$, then $\mathscr{M}$ is a subspace of $H^2(\sigma)$ which is invariant under $H^\infty(\sigma)$ but certainly is not of the form $\chi_F L^2(\sigma)$ for any measurable set F because $H^2(\sigma)$ contains no nonconstant real-valued functions. But $\chi_{X-E}\mathscr{M}=\{0\}\subseteq\mathscr{M}$ and χ_{X-E} is not in $H^\infty(\sigma)$. Thus, by [**1**, Chapter VII, Lemma 8.1], $H^\infty(\sigma)$ is not a maximal weak-* closed subalgebra of $L^\infty(\sigma)$. Thus (3) implies (2).

Clearly (2) implies (1), but it is not so clear that the converse is true. After all, $H^\infty(\sigma)$ would be an integral domain if it were the case that no two functions in $H^\infty(\sigma)$ are supported on disjoint sets.

PROOF THAT (1) IMPLIES (3). Suppose $H^\infty(\sigma)$ is not a maximal weak-* closed subalgebra of $L^\infty(\sigma)$ and let B be a proper weak-* closed subalgebra of $L^\infty(\sigma)$ which contains $H^\infty(\sigma)$ properly. We prove the following assertion about B.

(4) The $L^2(\sigma)$-closure of B, $[B]_2$, is different from $L^2(\sigma)$.

Since $B\neq L^\infty(\sigma)$ and since B is weak-* closed, there is a function f in $L^1(\sigma)$ which annihilates B. Since $H^\infty(\sigma)\subseteq B$, f annihilates $H^\infty(\sigma)$ and so f lies in $\bar{H}_0^1(\sigma)$ [**4**, Theorem 2.3.8] where for all p, $1\leqq p\leqq\infty$, $H_0^p(\sigma)=\{f\in H^p(\sigma)\mid \int_X f\,d\sigma=0\}$. If $k=\min\{1/|f|, 1\}$, then k is in $L^\infty(\sigma)$ and $\log k$ is in $L^1(\sigma)$. Consequently, by [**4**, Theorem 2.5.9] there is a ψ in $H^\infty(\sigma)$ such that $|\psi|=k$ a.e. It follows easily that $\bar{\psi}f$ is in $L^2(\sigma)$ and is orthogonal to $[B]_2$. Thus (4) is proved.

Let $\mathscr{K}=L^2(\sigma)\ominus[B]_2$ so that, by (4), $\mathscr{K}\neq\{0\}$. Since B contains $H^\infty(\sigma)$, $\mathscr{K}$ is orthogonal to $H^2(\sigma)$ and therefore, $\mathscr{K}\subseteq\bar{H}_0^2(\sigma)$ [**4**, p. 226]. Let $\mathscr{S}_B=\{E\in\mathscr{S}\mid\chi_E\in B\}$. We make the following assertions about $\mathscr{S}_B$.

(5) $\mathscr{S}_B$ is a subalgebra of $\mathscr{S}$.

(6) For all E in $\mathscr{S}_B$, $\chi_E\mathscr{K}\subseteq\mathscr{K}$.

(7) There is an E in $\mathscr{S}_B$ such that $0<\sigma(E)<1$.

Assertion (5) is clear and so is (6). For if $E\in\mathscr{S}_B$, then $\chi_E\in B$ so $\chi_E[B]_2\subseteq[B]_2$. Since as an operator on $L^2(\sigma)$, χ_E is Hermitian, we see that $\chi_E\mathscr{K}\subseteq\mathscr{K}$. To prove (7), choose a φ in B which is invertible in B but which does not belong to $H^\infty(\sigma)$. Such a choice is possible since a Banach algebra with identity is spanned by its invertible elements and since $B\neq H^\infty(\sigma)$. Since φ is invertible in B, it is invertible in $L^\infty(\sigma)$, and so, by [**4**, Corollary 4.1.5], there is a unimodular function θ in $L^\infty(\sigma)$ and a function ψ in $H^\infty(\sigma)$ which is invertible in $H^\infty(\sigma)$ such that $\varphi=\theta\psi$. Since φ does not belong to $H^\infty(\sigma)$, θ is not constant. Moreover, since φ and ψ are invertible in B, θ and $\theta^{-1}=\bar{\theta}$ lie in B. As an algebra of operators on $L^2(\sigma)$, B is weakly closed and since B contains the unitary operator θ and its adjoint, it contains all the spectral projections of θ. These are of the form χ_E for measurable sets E. Since as a function θ is not constant, as a unitary operator θ has spectral projections different from zero and the identity. Thus (7) is proved.

We now consider two mutually exclusive and exhaustive cases which may occur.

Case I. There is an E in $\mathscr{S}_B$ such that $\{0\}\neq\chi_E\mathscr{K}\neq\mathscr{K}$. In this case, there are nonzero functions in $\mathscr{K}$ with disjoint supports. Since $\overline{\mathscr{K}}\subseteq H^2(\sigma)$, there are nonzero functions in $H^2(\sigma)$ with disjoint supports. This implies that there are nonzero functions in $H^\infty(\sigma)$ with disjoint supports. For, if f is any function in $H^p(\sigma)$, $1\leqq p<\infty$, then there is a function g in $H^\infty(\sigma)$ whose support is that of f. To see this simply choose a function ψ in $H^\infty(\sigma)$ whose modulus is the minimum of $1/|f|$ and 1 (we remarked earier that such a choice is always possible) and let $g=\psi f$. Then g has the desired properties. Thus if Case I holds, $H^\infty(\sigma)$ is not an integral domain.

Case II. For all E in $\mathscr{S}_B$, either $\chi_E\mathscr{K}=\{0\}$ or $\chi_E\mathscr{K}=\mathscr{K}$. Let $\mathscr{T}=\{E\in\mathscr{S}_B|\chi_E\mathscr{K}=\mathscr{K}\}$. Then $\mathscr{T}$ is nonempty and contains sets with measure different from zero and one. Direct $\mathscr{T}$ by inclusion and consider the decreasing net of projections $\{\chi_E\}_{E\in\mathscr{T}}$. This net converges in the strong operator topology to its greatest lower bound which is a projection χ_{E_0}. Since B is weakly closed as an algebra of operators on $L^2(\sigma)$, χ_{E_0} lies in B. Since $\chi_E\mathscr{K}=\mathscr{K}$ for all E in $\mathscr{T}$, $\chi_{E_0}\mathscr{K}=\mathscr{K}$; i.e., E_0 lies in $\mathscr{T}$. Finally, since $\mathscr{T}$ contains sets of measure different from zero and one, $0<\sigma(E_0)<1$. It is easy to see that χ_{E_0} is a minimal projection in B in the sense that if χ_F is in B and if $F\subseteq E_0$, then either $\chi_{E_0}=\chi_F$ or $\chi_F=0$. For, if F is a set in $\mathscr{S}_B$ continued in E_0, then either $\chi_F\mathscr{K}=\mathscr{K}$ or $\chi_F\mathscr{K}=\{0\}$ by hypothesis. In the first case F lies in $\mathscr{T}$ so $\chi_F=\chi_{E_0}$, while in the second χ_{E_0-F} is in $\mathscr{T}$, $\chi_{E_0-F}=\chi_{E_0}$, and $\chi_F=0$.

Since $\chi_{E_0}\mathscr{K}=\mathscr{K}$, every function in $\mathscr{K}$ is supported on E_0 and therefore, by an argument given above, there are functions in $H^\infty(\sigma)$ supported on E_0. The proof will be completed by showing that there are nonzero functions in $H^\infty(\sigma)$ which are supported on $X-E_0$.

To this end, consider the function $\psi=\exp(\chi_{X-E_0})$ which is an invertible element of B and which does not belong to $H^\infty(\sigma)$. By [**4**, Corollary 4.1.5], there is a function φ which is invertible in $H^\infty(\sigma)$ and a unimodular function θ such that $\psi=\theta\varphi$. Since ψ does not belong to $H^\infty(\sigma)$, θ is not constant. As before, both θ and θ^{-1} lie in B and so B contains all the spectral projections of θ regarded as a unitary operator on $L^2(\sigma)$. Since as a function θ is not constant, as a unitary operator θ has spectral projections different from zero and the identity. Moreover, since χ_{E_0} is a minimal projection in B, the unitary operator θ must have an eigenvalue α ($|\alpha|=1$) with eigenspace $\chi_F L^2(\sigma)$ containing $\chi_{E_0} L^2(\sigma)$. Consequently, we we may write the equation

$$\theta = \alpha\chi_F + \chi_{X-F}\theta.$$

From this, the following equation is valid for all x in E_0:

$$1 = \psi(x) = [\alpha\chi_F(x) + \chi_{X-F}(x)\theta(x)]\varphi(x) = \alpha\varphi(x).$$

Thus for all x in E_0, $\varphi(x)=\bar{\alpha}$. Whence, the function $\varphi-\bar{\alpha}$ lies in $H^\infty(\sigma)$, is supported on $X-E_0$, and is clearly nonzero. Thus in Case II, $H^\infty(\sigma)$ is also not an integral domain and the proof of the theorem is complete.

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