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CONDITION FOR A FUNCTION SPACE TO BE LOCALLY COMPACT

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ABSTRACT. Let F be an equicontinuous family of functions from a compact Hausdorff space to a locally compact Hausdorff uniform space. In this paper we prove that the pointwise closure of F is locally compact relative to the topology of uniform convergence.

A function space is proved *compact* by identifying it with a closed subset of a (in general) nonfinite product of compact spaces. Since a nonfinite product of locally compact spaces is not necessarily locally compact this approach is not suitable to prove a function space *locally compact*. Instead to obtain the present result we identify F, the family of functions from X to Y, with a closed subset of the hyperspace of compact subsets of $X \times Y$ endowed with the finite topology.

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Throughout, F is a family of functions from a topological space X to a topological or uniform space Y, f_α is a net of functions from X to Y, and f is a function from X to Y. We identify a function f from X to Y with its graph $\{(x, f(x)) : x \in X\}$ and thus consider it a subset of $X \times Y$.

We refer the reader to Kelley [**1**] for standard definitions and results not given here.

If Z is a set and W is a subset of Z, $P(W)$ is the collection of all nonempty subsets of W and $R(W, Z)$ is the collection of all nonempty subsets of Z which intersect W.

DEFINITIONS. Let Z be a topological space and W any open subset of Z. The family of all sets of the form $P(W)$ is a basis for the *upper semifinite* (*usf*) topology on $P(Z)$. The family of all sets of the form $R(W, Z)$ is a subbasis for the *lower semifinite* (*lsf*) topology on $P(Z)$. The *finite* topology on $P(Z)$ is the least upper bound of the usf and lsf topologies. Of course

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we may relativise these topologies to subcollections of $P(Z)$ such as $C(Z)$ the collection of all nonempty compact subsets of Z.

PROPOSITION 1. *The lsf topology on F is contained in the topology of pointwise convergence. If $(Y, \mathscr{V})$ is a uniform space and F is equicontinuous the two topologies coincide on F.*

PROOF. Let f be in F and f_α be a net in F which converges pointwise to f. Let $R(U\times V, X\times Y)$ be a subbasic neighborhood of f in the lsf topology and $(p, f(p))$ be in $f\cap U\times V$ where U and V are open. Then $f(p)$ is in V, so $f_\alpha(p)$ is eventually in V. Therefore $(p, f_\alpha(p))$ is eventually in $U\times V$. It follows that f_α converges to f in the lsf topology, so the lsf topology is contained in the pointwise topology.

If $(Y, \mathscr{V})$ is a uniform space and F is equicontinuous, let f be in F and f_α be a net in F which converges to f in the lsf topology. Let p be in X and V in $\mathscr{V}$. There is an open symmetric V_1 in $\mathscr{V}$ so that $V_1^2\subset V$. Since F is equicontinuous there is a neighborhood U of p so that x in U implies $(f_\alpha(x), f_\alpha(p))$ is in V_1 for all α. Since f_α converges to f in the lsf topology there is an α_0 so that $\alpha\geqq\alpha_0$ implies $f_\alpha\cap(U\times V_1[f(p)])\neq\varnothing$.

Hence for $\alpha\geqq\alpha_0$ there is an x_α in U with $f_\alpha(x_\alpha)$ in $V_1[f(p)]$. Accordingly, for $\alpha\geqq\alpha_0$, $(f_\alpha(p), f(p))$ is in $V_1^2\subset V$. Therefore, f_α converges to f pointwise.

Thus on F the lsf and pointwise topologies coincide.

PROPOSITION 2. *Let X be compact Hausdorff, Y be a uniform space, and F be a family of continuous functions. On F, the topology of uniform convergence and the usf topology coincide.*

PROOF. This is essentially Theorem 4.2 or 4.6 of Naimpally [3]. Note that on F the graph topology and usf topology are the same.

PROPOSITION 3. *Let F be an equicontinuous family of functions and $(Y, \mathscr{V})$ a Hausdorff uniform space. Let $\hat{F}$ denote the pointwise closure of F in Y^X. Then $\hat{F}$ is closed in the finite topology on $P(X\times Y)$.*

PROOF. Let f_α be a net in $\hat{F}$ which converges to A in $P(X\times Y)$ relative to the finite topology.

First we show that A is a function from a subset of X to Y. Suppose for some p in X, (p, q_1) and (p, q_2) belong to A where $q_1\neq q_2$. There exists a symmetric V in $\mathscr{V}$ so that $q_1\notin V^4(q_2)$. Since F is equicontinuous, a neighborhood U of p exists so that x in U implies $(f_\alpha(x), f_\alpha(p))$ is in V for all α.

Since f_α converges to A in the lsf topology there is a β so that $f_\beta\cap(U\times V[q_i])\neq\varnothing$ for $i=1, 2$. Thus for $i=1, 2$ there is an x_i in U so that $f_\beta(x_i)\in V[q_i]$.

Consequently we have in summary that $(q_1, f_\beta(x_1))$, $(f_\beta(x_1), f_\beta(p))$, $(f_\beta(p), f_\beta(x_2))$ and $(f_\beta(x_2), q_2)$ are in V so that q_1 is in $V^4[q_2]$. This contradiction shows that A is a function.

Now we show that the domain of A is X. Suppose $p \notin \operatorname{dom} A$. Since Y is Hausdorff, $\{p\}$ is closed. Thus $\{p\} \times Y$ is a closed set which does not intersect A. Then we see that $X \times Y - \{p\} \times Y$ is a neighborhood of A in the usf topology on $P(X \times Y)$, but f_α (since $\operatorname{dom} f_\alpha = X$) cannot be eventually in this neighborhood.

Thus A is in Y^X and so applying Proposition 1 to $\hat{F} \cup \{A\}$ we have that f_α converges pointwise to A. Therefore A is in $\hat{F}$, and $\hat{F}$ is closed in $P(X \times Y)$ with the finite topology.

THEOREM. *Let F be an equicontinuous family of functions from a compact Hausdorff space X to a locally compact Hausdorff uniform space Y. Let $\hat{F}$ be the pointwise closure of F. Then $\hat{F}$ is locally compact in the topology of uniform convergence.*

PROOF. Under our hypothesis pointwise convergence is equivalent to uniform convergence on $\hat{F}$. Thus by Propositions 1 and 2 the topology of uniform convergence and the finite topology coincide on $\hat{F}$.

Now all members of $\hat{F}$ are compact in $X \times Y$ since they are continuous and X is compact, so $\hat{F} \subset C(X \times Y)$. By Proposition 3, $\hat{F}$ is closed in $P(X \times Y)$ with the finite topology and thus since $\hat{F} \cap C(X \times Y) = \hat{F}$, $\hat{F}$ is closed in $C(X \times Y)$.

Since $X \times Y$ is locally compact Hausdorff, $C(X \times Y)$ is locally compact (see Michael [**2**, Proposition 4.4.1, p. 162]; note "2^X" should be "$C(X)$"). Therefore since $\hat{F}$ is a closed subset of a locally compact space, it is locally compact itself.

REMARK. Professor J. S. Yang points out that the statement obtained from the preceding theorem by letting X be locally compact instead of compact and replacing uniform convergence by uniform convergence on compacta is false. Let X be an uncountable set with the discrete topology and Y be the reals. Then Y^X is equicontinuous but is not locally compact in the topology of uniform convergence on compacta since this coincides with the topology of pointwise convergence.

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