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ON THE ARENS PRODUCT AND COMMUTATIVE BANACH ALGEBRAS

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ABSTRACT. The purpose of this note is to generalize two recent results by the author for commutative Banach algebras. Let A be a commutative Banach algebra with carrier space X_A and π the canonical embedding of A into its second conjugate space A^{**} (with the Arens product). We show that if A is a semisimple annihilator algebra, then $\pi(A)$ is a two-sided ideal of A^{**}. We also obtain that if A is a dense two-sided ideal of $C_0(X_A)$, then $\pi(A)$ is a two-sided ideal of A^{**} if and only if A is a modular annihilator algebra.

1. **Notation and preliminaries.** Notation and definition not explicitly given are taken from Rickart's book [5].

For any subset E of a Banach algebra A, let $L_A(E)$ and $R_A(E)$ denote the left and right annihilators of E in A, respectively. Then A is called a modular annihilator algebra if, for every maximal modular left ideal I and for every maximal modular right ideal J, we have $R_A(I)=(0)$ if and only if $I=A$ and $L_A(J)=(0)$ if and only if $J=A$ (see [**2**, p. 568, Definition]).

Let A be a Banach algebra, A^* and A^{**} the conjugate and second conjugate spaces of A, respectively. The Arens product on A^{**} is defined in stages according to the following rules (see [**1**]). Let $x, y \in A, f \in A^*$, F, $G \in A^{**}$.

(a) Define $f \circ x$ by $(f \circ x)(y)=f(xy)$. Then $f \circ x \in A^*$.

(b) Define $G \circ f$ by $(G \circ f)(x)=G(f \circ x)$. Then $G \circ f \in A^*$.

(c) Define $F \circ G$ by $(F \circ G)(f)=F(G \circ f)$. Then $F \circ G \in A^{**}$.

A^{**} with the Arens product $\circ$ is denoted by $(A^{**}, \circ)$. Let π be the canonical embedding of A into A^{**}. Then $\pi(A)$ is a subalgebra of $(A^{**}, \circ)$.

Let A be a Banach algebra. For each element $x \in A$, let $\mathrm{Sp}_A(X)$ denote the spectrum of x in A. If A is commutative, X_A will denote the carrier space of A and $C_0(X_A)$ the algebra of all complex-valued continuous functions on X_A, which vanish at infinity; $C_0(X_A)$ is a commutative B^*-algebra.

In this paper, all algebras and linear spaces under consideration are over the complex field C.

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2. **The results.** Our first result is a generalization of [7, p. 82, Theorem 3.3] for commutative Banach algebras.

THEOREM 2.1. *Let A be a semisimple commutative annihilator Banach algebra. Then A is a two-sided ideal of $(A^{**}, \circ)$.*

PROOF. Let X_A be the carrier space of A and let $B=C_0(X_A)$. Since A is an annihilator algebra, it is well known that X_A is discrete and therefore B is a dual B^*-algebra by [**6**, p. 532, Theorem 4.2]. Let $|\cdot|$ be the norm on B. Then the given norm $\|\cdot\|$ majorizes $|\cdot|$ on A. Considering A as a subalgebra of B, we show that A is dense in B. Let $x\neq 0$ be a hermitian element in B. Then it is known that $\mathrm{Sp}_B(x)$ has no nonzero limit points and so it is a countable set (see [**8**, p. 826, Theorem 3.1]). Therefore it follows from [**5**, p. 111, Theorem (3.1.6)] that $\{\alpha(x):\alpha \in X_A\}$ is countable. Denote those $\alpha \in X_A$ for which $\alpha(x)\neq 0$ by $\alpha_1, \alpha_2, \cdots$. Then $\alpha_n(x)\to 0$ as $n\to\infty$. For each α_n, by Šilov's theorem [**5**, p. 168, Theorem (3.6.3)], there exists a nonzero idempotent e_n in A such that $e_n(\alpha_n)=1$ and $e_n(\alpha)=0$ for all $\alpha\neq\alpha_n$.

Let $\varepsilon>0$ be given. Then there exists a positive integer N such that $|\alpha_n(x)|<\varepsilon$ for all $n\geqq N$. Let $y=\sum_{n=1}^{N} \alpha_n(x)e_n$. Then $y \in A$ and it is easy to see that

$$|x-y| = \sup\{|\alpha(x)-\alpha(y)|:\alpha \in X_A\} < \varepsilon.$$

Therefore it follows now that A is dense in B. Since A is a dual B^*-algebra, by [7, p. 82, Theorem 3.3], B is a two-sided ideal of B^{**} (with the Arens product). Therefore, by [7, p. 82, Lemma 3.2], A is a two-sided ideal of $(A^{**}, \circ)$ and the proof is complete.

The following result is a generalization of [**8**, p. 830, Theorem 5.2] for commutative Banach algebras.

THEOREM 2.2. *Let A be a commutative Banach algebra such that A is a dense two-sided ideal of $C_0(X_A)$. Then A is a two-sided ideal of $(A^{**}, \circ)$ if and only if A is a modular annihilator algebra.*

PROOF. Let $B=C_0(X_A)$ and let $|\cdot|$ be the norm on B. By [**3**, p. 3, Theorem 2.3], there exists a constant k such that $\|ab\|\leqq k\|a\|\,|b|$ and $\|ab\|\leqq k|a|\,\|b\|$ for all $a, b \in A$. It is easy to see that A is a semisimple algebra. Suppose A is a two-sided ideal of $(A^{**}, \circ)$. Then by the proof of [**8**, p. 829, Lemma 5.1], we can show that X_A is discrete and therefore A is a modular annihilator algebra (see [**2**, p. 578, Example 8.4]). Conversely suppose A is a modular annihilator algebra. Then, by [**2**, p. 569, Theorem 4.2 (6)], X_A is discrete in the hull-kernel topology and therefore X_A is discrete in the finer Gelfand topology. Hence B is a dual B^*-algebra. Now by the argument in [**8**, p. 830, Theorem 5.2], we can show that A is a two-sided ideal of $(A^{**}, \circ)$, and this completes the proof.

COROLLARY 2.3. *Let A be as in Theorem 2.2. If A is reflexive and has an approximate identity, then A is finite dimensional.*

PROOF. By [**4**, p. 855, Lemma 3.8], A has an identity element. Also it follows from Theorem 3.2 that A is a modular annihilator algebra and therefore it is finite dimensional by [**2**, p. 573, Proposition 6.3].

Let G be a compact abelian group with the Haar measure and let $A=L_2(G)$. Then it is well known that A is reflexive and A is a dense two-sided ideal of $C_0(X_A)$. Also if A is infinite dimensional, A has no approximate identity.

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