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INVERTING SETS FOR FUNCTION ALGEBRAS

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ABSTRACT. If A is a function algebra on X, then we say that X is an inverting set for A if $f \in A$ and f does not vanish on X implies f is invertible in A. We obtain results on inverting sets for tensor products and for extensions of $R(X)$ by real valued functions.

1. **Introduction.** Let X be a compact Hausdorff space and let $C(X)$ denote the algebra of complex valued continuous functions on X with supremum norm. Let A be a function algebra on X. We say that X is an *inverting set* for A if $f \in A$ and f does not vanish on X implies $1/f \in A$. We denote the Šilov boundary of A by ∂_A where $\partial_A \subseteq X$. We say that X is the maximal ideal space of A and write $M(A)=X$ if for each nonzero multiplicative linear functional ϕ on A there is $x \in X$ satisfying $\phi(f)=f(x)$ for each $f \in A$.

Let $\Delta=\{z \in C:|z|\leqq 1\}$ and let $P(\Delta\times\Delta)$ denote the uniform closure in $C(\Delta\times\Delta)$ of the polynomials in z and w. Then $\{(z, w) \in \Delta\times\Delta:|z|=1$ or $|w|=1\}$ and $\{(z, w):|z|=|w|\leqq 1\}$ are inverting sets for $P(\Delta\times\Delta)$. We generalize this result to tensor products of function algebras. Let X be a compact subset of C and let $R(X)$ denote the uniform closure in $C(X)$ of the rational functions with poles off X. If $\mathfrak{F}$ is a set of continuous real valued functions on X, then X is an inverting set for the closed algebra generated by $R(X)$ and $\mathfrak{F}$. Finally, we give an example of a function algebra contained in the disc algebra for which Δ is not an inverting set.

2. **Tensor products.** Let A and B be function algebras on X and Y respectively. Let $A\otimes B$ denote the closed linear span in $C(X\times Y)$ of functions of the form $f\otimes g$ where $f \in A$ and $g \in B$ and $(f\otimes g)(x, y)=f(x)g(y)$. We next give two results which are generalizations to tensor products of the behavior of zero sets of functions in the bidisc algebra.

THEOREM 1. *Suppose A and B are function algebras on $X=M(A)$ and $Y=M(B)$ respectively. Then $(\partial_A\times Y)\cup(X\times\partial_B)$ is an inverting set for $A\otimes B$.*

PROOF. Fix $F \in A\otimes B$ and assume F does not vanish on $\partial_A\times Y$ or $X\times\partial_B$. Let $K=\{x \in X: F(x, -)$ is invertible in $B\}$. Notice that $K\supset\partial_A$ and

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that K is open in X. There exists $\varepsilon>0$ such that $|F(x, y)|\geqq\varepsilon$ for each $x\in X$ and $y\in\partial_B$. Suppose $x_\alpha\in K$ and $x_\alpha\to x_0$. Then $|F(x_\alpha, y)|\geqq\varepsilon$ for each $y\in Y$ so $|F(x_0, y)|\geqq\varepsilon$ for each $y\in Y$. Thus K is closed and we apply the Šilov idempotent theorem to obtain $K=X$. Hence, F is invertible in $A\otimes B$ since $M(A\otimes B)=X\times Y$.

THEOREM 2. *Suppose A and B are function algebras on $X=M(A)$ and $Y=M(B)$ respectively. Suppose $f\in A$ and $g\in B$ satisfy $|f(x)|=|g(y)|=1$ for each $x\in\partial_A$ and $y\in\partial_B$ and $f(X)=g(Y)=\Delta$. Then $\{(x,y):|f(x)|=|g(y)|\}$ is an inverting set for $A\otimes B$.*

PROOF. Suppose $F\in A\otimes B$ and F does not vanish on $\{(x,y):|f(x)|=|g(y)|\}$. We may assume that A and B are antisymmetric. Let $K=\{x\in X: F(x,y)\neq 0 \text{ if } |g(y)|\leqq|f(x)|\}$. It suffices to show $K=X$. By the Šilov idempotent theorem, we only need to show that K is open, closed and nonempty. Notice that K is nonempty since there is $x\in X$ such that $f(x)=0$. Clearly, $X\backslash K$ is closed. Suppose $x_\alpha\in K$ and $x_\alpha\to x_0$. Set $r=|f(x_0)|$. There is $\varepsilon>0$ such that $|F(x_0, y)|>\varepsilon$ for each y satisfying $|g(y)|=r$. Hence, we may assume, by taking a subnet, that $|F(x_\alpha, y)|\geqq\varepsilon$ for each y satisfying $|g(y)|=r$. Let $Z=\{y\in Y:|g(y)|\leqq r\}$. Then B_z, the closure of $B|Z$, has Z as its maximal ideal space. By the local maximum modulus theorem, the Šilov boundary of B_Z is contained in $\{y:|g(y)|=r\}$. Hence, $x_\alpha\in K$ gives $|F(x_\alpha, y)|\geqq\varepsilon$ for each $y\in Z$. Passing to the limit, we have $|F(x_0, y)|\geqq\varepsilon$ for each $y\in Z$ and so $x_0\in K$. Thus, K is open, closed and nonempty.

Note. The Šilov boundary of $P(\Delta\times\Delta)$ is not an inverting set but is the intersection of two inverting sets: $\{(z, w)\in\Delta\times\Delta:|z|=1 \text{ or } |w|=1\}$ and $\{(z, w):|z|=|w|\leqq 1\}$.

3. **Extensions of** $R(X)$. If X is a compact subset of $\mathbf{C}$, then $P(X)$ denotes the uniform closure in $C(X)$ of the polynomials in z, $R(X)$ denotes the uniform closure in $C(X)$ of the rational functions with poles off X, and $A(X)$ denotes the algebra of continuous functions on X which are analytic in the interior of X. If A is a function algebra on X and if $\mathfrak{F}$ is a set of continuous functions on X, then $A[\mathfrak{F}]$ denotes the closed algebra generated by A and $\mathfrak{F}$. We prove the following theorem in this section.

THEOREM 3. *Let X be a compact subset of $\mathbf{C}$. Suppose $A=R(X)$ or $A=A(X)$ and let $\mathfrak{F}$ be a set of real valued continuous functions on X. Then X is an inverting set for $A[\mathfrak{F}]$.*

We first prove the following lemma concerning the zeros of functions in $A(X)$.

LEMMA. *Let X be a compact subset of C. Suppose $f \in A(X)$ and $g \in C(X)$ such that g does not vanish on X and $|f-g|<|g|$ on ∂X, the boundary of X. Then f does not vanish on X.*

PROOF. We may assume that g is a polynomial in x and y. Let $K=\{x \in X:|f(x)-g(x)| \geqq |g(x)|\}$. Then K is a compact subset of X°, the interior of X. Let G be a smoothly bounded open set such that $K \subseteq G$ and $\bar{G} \subseteq X^\circ$. Since $|f/g-1|<1$ on ∂G, there exists a smooth function h on ∂G such that $f=ge^h$ on ∂G. Set n equal to the number of zeros of f in G and notice $df/f=dg/g+dh$. Apply Green's theorem and the argument principle to obtain

$$2\pi i \cdot n = \int_{\partial G} \frac{df}{f} = \int_{\partial G} \frac{dg}{g} + \int_{\partial G} dh = \iint_G d\left(\frac{dg}{g}\right) = 0.$$

We conclude that f does not vanish on X.

PROOF OF THEOREM 3. Let $f \in A[\mathfrak{F}]$ and assume that f does not vanish on X. Let K be a maximal set of antisymmetry for $A[\mathfrak{F}]$. To show that f is invertible in $A[\mathfrak{F}]$, we only need to show that $f|K$ is invertible in $A[\mathfrak{F}]|K$ by Bishop's antisymmetric decomposition theorem [**1**, p. 60]. Let Y be the A-convex hull of K in X. Let B denote the closure of $A|Y$. Thus, $M(B)=Y$ and $B|K=A[\mathfrak{F}]|K$ since each member of $\mathfrak{F}$ is constant on K and K is a peak set for $A[\mathfrak{F}]$. Let g be a member of B satisfying $f=g$ on K. Since f does not vanish on Y and $K \supset \partial Y$, the Lemma implies that g does not vanish on Y and hence g is invertible in B. We conclude that $f|K$ is invertible in $A[\mathfrak{F}]|K=B|K$.

4. **Examples.** In this section we give four examples concerning inverting sets. We begin by giving examples to show that Theorem 3 cannot be generalized to the case where $\mathfrak{F}$ is a single function of constant modulus or where the space X is a polynomially convex subset of C^2. We give an example of a function algebra on a Cantor set K such that K is not an inverting set. We use this algebra to construct a function algebra B contained in the disc algebra such that the closed unit disc is not an inverting set for B.

EXAMPLE 1. Let $A=P(\Delta)[f]$ where $f(z)=e^{i2\pi|z|}$. This algebra was considered by Wilken [**3**, Example 3.1]. The function f is not invertible in A and so Δ is not an inverting set for A. One can see this by noticing that

$$\phi(g) = \int_0^1 \frac{1}{2\pi} \int_{-\pi}^{\pi} g(re^{i\theta})\, d\theta\, dr$$

is a multiplicative linear functional on A and $\phi(f)=0$.

EXAMPLE 2. Let $X=\{(z, e^{i2\pi|z|}):|z|\leqq 1\}$ and let $\mathfrak{F}=\{f \in C(\Delta\times\Delta):f$ is real valued and $f=0$ on $X\}$. Then $\Delta\times\Delta$ is not an inverting set for $P(\Delta\times\Delta)[\mathfrak{F}]$. Define g on $\Delta\times\Delta$ by $g(z, w)=i2\pi|z|$. Then e^g belongs to A but, applying remarks in Example 1, one sees that e^g is not invertible in A.

EXAMPLE 3. We construct a function algebra A on a Cantor set K such that K is not an inverting set for A. Let X denote the maximal ideal space of $L^\infty(T, m)$ where T is the unit circle and m is Lebesgue measure. We consider H^∞, the algebra of bounded analytic functions on the open unit disc, as a subalgebra of L^∞ and we denote the Gelfand map of L^∞ onto $C(X)$ by $f\to\hat{f}$. If g is a continuous function on X and if $g>0$, then there is a bounded real valued Borel function u on T such that $\hat{u}=\log g$ and the function

$$h(z) = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}u(e^{i\theta})\,d\theta\right]$$

belongs to H^∞ and satisfies the conditions $|h|=e^u$ and $|\hat{h}|=g$ (see [**2**, p. 182]).

Let $f_0=z$ and $d_0(x, y)=|\hat{z}(x)-\hat{z}(y)|$ for $x, y \in X$. Then d_0 is a pseudometric on X. Since X is totally disconnected, there exist $f_1, f_2, \cdots$ in H^∞ such that $|\hat{f}_n|$ has finite range and the pseudometric d_n on X, defined by

$$d_n(x, y) = \sum_{k=0}^{n} 2^{-k}\,|\hat{f}_k(x)-\hat{f}_k(y)|,$$

has the property that each component of (X, d_n) has d_n-diameter less than $1/n$. Set $d(x, y)=\lim_{h\to\infty} d_n(x, y)$. Identifying points in X which are d-distance zero apart, one obtains a compact metric space Y such that Y is totally disconnected. Let A be the function algebra on Y generated by the functions $\hat{f}_0, \hat{f}_1, \cdots$ which can be considered as defined on Y. Then Y is not an inverting set since $\hat{f}_0=z$ maps Y onto T and z is not invertible in H^∞. Now, set $K=\partial_A$. Finally, K is a Cantor set since $K\subseteq Y$ and K has no isolated points. If y is an isolated point in K, then $\chi_{\{y\}}$ belongs to A but H^∞ has no idempotents except 0 and 1.

EXAMPLE 4. Using Example 3, one may obtain a function algebra B which is contained in the disc algebra such that Δ is not an inverting set for B. Let A and K be as in Example 3. Let K' be a Cantor set of measure zero in T and let ψ be a homeomorphism of K onto K'. Let

$$B = \{f \in P(\Delta):f\circ\psi \in A\}.$$

Then B is a function algebra on Δ since K' is a peak interpolation set for $P(\Delta)$. Choose $f \in A$ such that f is not invertible in A but f does not vanish on K. Since K is a Cantor set, there is a continuous function g on K such that $e^g=f$. Choose $\bar{g} \in P(\Delta)$ such that $\bar{g}\circ\psi=g$. Then $\exp(\bar{g})$ belongs to B

and is not invertible in B since $\exp(\bar{g} \circ \psi) = f$. Of course $\exp(\bar{g})$ does not vanish on Δ.

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