PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 1, January 1973

PD-MINIMAL SOLUTIONS OF $\Delta u=Pu$ ON OPEN RIEMANN SURFACES

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ABSTRACT. By means of the Royden compactification of an open Riemann surface R necessary and sufficient conditions are given for a Dirichlet-finite solution of $\Delta u=Pu$ ($P\geqq 0$, $P\not\equiv 0$) to be PD-minimal on R. A relation between PD-minimal solutions and HD-minimal solutions is obtained. In addition it is shown that the dimension of the space of PD-solutions is the same as the number of P-energy nondensity points in the finite dimensional case.

Let $P(z)\,dx\,dy$ ($z=x+iy$), $P\not\equiv 0$, be a nonnegative C^1 differential on an open Riemann surface R. Denote by $PD(R)$ the Hilbert space of all Dirichlet-finite solutions of the second-order selfadjoint elliptic partial differential equation

$$\Delta u(z) = P(z)u(z) \tag{1}$$

on R where $\Delta u(z)dxdy=d^*\,du(z)$. The scalar product is given by $(u, v)=D_R(u, v)=\int_R du\wedge {}^*dv$, *not* the energy integral $E_R(u, v)=D_R(u, v)+\int_R P^2uv$. Observe that the only constant solution of (1) is the identically zero solution. The classification problem with respect to $\Delta u=Pu$ was initiated by Ozawa [9] who investigated the class $PE(R)$ of energy-finite solutions of (1) on R. The class $PD(R)$ itself was first considered by Royden [**10**] in 1959. A little later the works of Nakai ([**5**], [**6**]) gave impetus to the theory of the class $PD(R)$. Recent contributions to the study of $PD(R)$ are contained in papers by Nakai ([7], [8]), Glasner-Katz [2], and Singer ([**12**], [**13**]).

The energy integral $E_R(u)\equiv E_R(u, u)$ plays the same role as the Dirichlet integral $D_R(u)\equiv D_R(u, u)$ in the harmonic case, i.e. solutions of $\Delta u=0$, and the class $PE(R)$ likewise shares many properties possessed by the class $HD(R)$ of Dirichlet-finite harmonic functions (see, for example, Ozawa [9], Glasner-Katz [2], Kwon-Sario-Schiff ([3], [4])). However, the class $PD(R)$ is quite different in nature from $HD(R)$. Nevertheless it does share

Received by the editors April 23, 1972.

AMS (MOS) subject classifications (1970). Primary 30A48, 31A05.

Key words and phrases. Royden harmonic boundary, P-energy nondensity point, harmonic projection, PD-minimal function, HD-minimal function, Riesz decomposition.

some common properties with $HD(R)$. For example, Nakai **[8]** has shown recently that the Virtanen identity $O_{HD}=O_{HBD}$ is also valid for $PD(R)$; namely, $O_{PD}=O_{PBD}$, where $PBD(R)$ is the class of bounded PD-functions on R.

The purpose of this paper is to give a necessary and sufficient condition for a PD-function to be PD-minimal. Although the statement itself is similar to that for HD-functions new techniques are required for the proofs. The most important tool used is the Royden harmonic boundary and in particular the subset Δ_p of P-energy nondensity points introduced by Nakai **[7]**. Further we give a relationship between HD-minimality and PD-minimality. Finally we also state a relation between the cardinality of Δ_p and the dimension of $PD(R)$ whenever the latter is finite. For the reader's convenience we shall briefly review some preliminaries in §1.

1. Let R^* be the *Royden compactification* of R (for details see e.g. Sario-Nakai **[11]**). Denote by $\Gamma=R^*-R$ the *Royden boundary* of R and by $\Delta=\Delta(R)$ the *Royden harmonic boundary* of R, consisting of points of Γ which are regular for the harmonic Dirichlet problem. A point Z^* in Δ will be called a *P-energy nondensity point* (cf. **[7]**) if there exists an open neighborhood U^* of Z^* in R^* such that

$$\int_{U\times U} G_U(z,\zeta)P(z)P(\zeta)\,dx\,dy\,d\xi\,d\eta < \infty \qquad (\zeta = \xi + i\eta) \tag{2}$$

for $z \in U$, where $U=U^*\cap R$ and G_U is the harmonic Green's function of U.

2. If R is parabolic then $PD(R)=\{0\}$ (cf. Royden **[10]**). We therefore assume throughout the paper that R is hyperbolic. Denote by $\tilde{M}(R)$ the class of all Dirichlet-finite Tonelli functions on R and by $\tilde{M}_\Delta(R)$ the subclass of $\tilde{M}(R)$ consisting of functions g such that $g|\Delta(R)=0$. We then have the orthogonal decomposition (cf. **[11]**):

$$\tilde{M}(R) = HD(R) + \tilde{M}_\Delta(R).$$

The subset $M(R)$ consisting of all bounded members of $\tilde{M}(R)$ is called the *Royden algebra* of R. It is known that $M(R)$ is closed under the lattice operations $f\cup g=\max(f,g)$, and $f\cap g=\min(f,g)$. Moreover $M(R)$ has the orthogonal decomposition

$$M(R) = HBD(R) + M_\Delta(R),$$

where $HBD(R)$ is the class of bounded harmonic functions on R and $M_\Delta(R)$ the subclass of $M(R)$ consisting of functions g with $g|\Delta=0$.

For each $f\in \tilde{M}(R)$ we denote by $\Pi_R f\in HD(R)$ the *harmonic projection* of f on R characterized by $f-\Pi_R f\in \tilde{M}_\Delta(R)$. Since $PD(R)\subset \tilde{M}(R)$ we may define the operator

$$\Pi_R \mid PD(R): PD(R) \to HD(R) \tag{3}$$

which is a vector space isomorphism from $PD(R)$ onto $\Pi_R(PD(R))$ such that $u>0$ is equivalent to $\Pi_R u>0$ and

$$\sup_R |u| = \sup_R |\Pi_R u| \qquad \text{(cf. [12])}. \tag{4}$$

Moreover it can be shown that if $u \in PD(R)$ then

$$u = \Pi_R u + T_R u \tag{5}$$

where

$$T_R u = -\frac{1}{2\pi}\int_R G_R(\cdot, \zeta)P(\zeta)u(\zeta)\, d\zeta\, d\eta \qquad (\zeta = \xi + i\eta)$$

and also

$$D_R(u) = D_R(\Pi_R u) + \frac{1}{2\pi}\int_{R\times R} G_R(z,\zeta)u(z)u(\zeta)P(z)P(\zeta)\, dx\, dy\, d\xi\, d\eta \tag{6}$$

(cf. [8]). If Ω is an open subset of R with smooth relative boundary $\partial\Omega$ (which may be empty in case $\Omega=R$) then for $u \in PD(\Omega)$ we obtain representations for u and $D_\Omega(u)$ as in (5), (6). Moreover

$$T_\Omega u \mid (\partial\Omega) \cup (\bar{\Omega} \cap \Delta) = 0, \tag{7}$$

where $\bar{\Omega}$ is the closure of Ω in R^*.

The following is an immediate consequence of the maximum principle for $PD(R)$ (cf. Glasner-Katz [2]):

LEMMA 1. *If $u \in PD(R)$ and $u|\Delta=0$ then $u\equiv 0$.*

3. Recall that Δ_p is the set of P-energy nondensity points of Δ. Now we state (cf. Nakai [7]):

LEMMA 2. *If $u \in PD(R)$ then $u|\Delta-\Delta_p=0$.*

PROOF. Let $Z_0 \in \Delta-\Delta_p$. Then for each neighborhood U^* of Z_0 in R^*,

$$\int_{U\times U} G_U(z,\zeta)P(z)P(\zeta)\, dx\, dy\, d\xi\, d\eta = \infty,$$

$U=U^* \cap R$. Suppose to the contrary that $u(Z_0)\neq 0$. Since each $u \in PD(R)$ possesses a Riesz decomposition (cf. [8]) as the difference of two nonnegative PD-functions on R we may assume that $u\geqq 0$ and $u(Z_0)>0$. Since u is continuous at z_0 there exists a neighborhood U^* of Z_0 in R^* such that

$u \geqq \delta > 0$ in U^*. But from (6) and the fact that $D_U(u) \leqq D_R(u) < \infty$ $(U = U^* \cap R)$ we have

$$D_U(u) = D_U(\Pi_U u) + \frac{1}{2\pi} \int_{U \times U} G_U(z, \zeta) u(z) u(\zeta) P(z) P(\zeta) < \infty,$$

which is impossible. Hence $u(Z_0) = 0$ as asserted.

COROLLARY 1. *If* $u \in PD(R)$ *and* $u|\Delta_p = 0$ *then* $u \equiv 0$.

The proof follows immediately from Lemmas 1 and 2.

4. A positive PD-function u on R which is not identically zero will be called *PD-minimal* if for any $v \in PD(R)$ such that $0 \leqq v \leqq u$ there exists a constant c_v such that $v = c_v u$ on R (for HD-minimal functions see Sario-Nakai **[11]**).

In contrast to HD-minimality which is characterized in terms of the entire harmonic boundary Δ, PD-minimality is stated solely in terms of Δ_p as follows:

THEOREM 1. *A PD-function on R is PD-minimal if and only if there exists an isolated point* $Z_0 \in \Delta_p$ *such that* $0 < u(Z_0)$ *and* $u = 0$ *on* $\Delta_p - \{Z_0\}$.

PROOF. We first establish the sufficiency. Since $\Delta = \Delta_p \cup (\Delta - \Delta_p)$ it follows from the hypothesis and Lemma 2 that $u|\Delta - \{Z_0\} = 0$. Now $\Pi_R u \in HD(R)$ by (3) and from (7) we deduce that $\Pi_R u(Z_0) = u(Z_0) > 0$ and $\Pi_R u = 0$ on $\Delta - \{Z_0\}$. Hence $\Pi_R u$ is HD-minimal, and in particular strictly positive and bounded (cf. **[11]**). From (4) it follows that u is bounded. For any $v \in PD(R)$ with $0 \leqq v \leqq u$ on R it follows from the continuity of PD-functions on Δ that $v = 0$ on $\Delta - \{Z_0\}$ and $0 \leqq v(Z_0) < \infty$. Hence $c_v u - v = 0$ on Δ where $c_v = v(Z_0)/u(Z_0)$. By Lemma 1, $v = c_v u$ on R and u is PD-minimal as was to be shown.

Conversely, assume that u is PD-minimal. Since $u \not\equiv 0$ by Corollary 1 there exists a point $Z_0 \in \Delta_p$ such that $u(Z_0) > 0$. There exists a neighborhood U^* of Z_0 as in (2). Suppose Z_0 is not an isolated point of Δ_p. Then consider any $Z_1 \in \Delta_p \cap U^*$ with $Z_1 \neq Z_0$. We claim that $u(Z_1) = 0$. Suppose to the contrary that $u(Z_1) > 0$. Note that we may assume that ∂U $(U = U^* \cap R)$ is smooth to begin with since we may modify U suitably otherwise. Select an $f \in M(U)$ such that $f(Z_0) = 1$, $f(Z_1) = 0$, $f|\partial U = 0$, and $0 \leqq f \leqq 1$ on U^*. Here $M(U)$ is the Royden algebra of bounded Dirichlet-finite Tonelli functions on U. Then $h = \Pi_U(f \cap u) \in HBD(U)$, $0 \leqq h \leqq u$ on U^*, $h|\partial U = 0$, $h(Z_1) = 0$, and $h(Z_0) = (f \cap u) Z_0$. Using the approach of Nakai [7] we now construct an appropriate $w \in PBD(R)$. We sketch the procedure here for the sake of completeness. By the method of exhaustion it is seen that the integral equation of the Fredholm type $(I - T_U)v = h$ has a unique solution $v \in PD(U)$, where I is the identity operator. Now $v|\partial U = 0$, $v(Z_0) = h(Z_0)$,

$v(Z_1)=0$, and $0 \leqq v \leqq h \leqq 1$. v is a Dirichlet-finite subsolution of (1). By the exhaustion method again, and by the weak Dirichlet principle (cf. **[8]**) we obtain a $w \in PBD(R)$ such that $v \leqq w \leqq 1$. Now $w|\Delta \cap U^* = v|\Delta \cap U^*$ by construction and $w=0$ on $\Delta \cap (R^* - U^*)$. Therefore $0 \leqq w \leqq u$ on Δ and hence on R. It follows that there is a constant c_w such that $w = c_w u$. But $w(Z_1) = 0 = c_w u(Z_1) > 0$, a contradiction. Hence $u(Z_1)=0$ as asserted.

Since u is continuous at Z_0 and $u(Z)=0$ for any $Z \neq Z_0 \in U^* \cap \Delta_p$ it follows that Z_0 is an isolated point of Δ_p.

To complete the proof we now show $u|\Delta_p - \{Z_0\} = 0$. Observe that for the function $w \in PBD(R)$ constructed above, $w(Z_0) = (f \cap u)Z_0$, $w|\Delta_p - \{Z_0\} = 0$ and $0 \leqq w \leqq u$ on Δ_p. Therefore $w = c_w u$ on R and so if there exists a $Z \in \Delta_p - \{Z_0\}$ such that $u(Z) > 0$ we obtain a contradiction $w(Z) = 0 = c_w u(Z) > 0$. This completes the proof.

COROLLARY 2. *If $Z \in \Delta_p$ is isolated in Δ_p then there always exists a $u \in PBD(R)$ such that $u(Z) > 0$ and $u=0$ on $\Delta - \{Z\}$. Also any PD-function v on R has a finite value at Z.*

For a proof of the second part we may assume $u \geqq 0$ on R since u has a Riesz decomposition. If $v(Z) = \infty$ then for $n = 1, 2, \cdots$ the inequality $nu \leqq v$ holds on Δ and hence on R. But this yields the contradiction $v \equiv \infty$.

5. A relation between *PD*-minimality and *HD*-minimality is given by

THEOREM 2. *If Π_R maps $PD(R)$ onto $HD(R)$ then $u \in PD(R)$ is PD-minimal if and only if $\Pi_R u \in HD(R)$ is HD-minimal.*

PROOF. First assume $u \in PD(R)$ is *PD*-minimal. Then for any $h \in HD(R)$ with $0 \leqq h \leqq \Pi_R u$ on R there exists a $v \in PD(R)$ such that $\Pi_R v = h$. From (5) and (7) we see that $u = \Pi_R u$ and $v = \Pi_R v$ on Δ. Hence $0 \leqq v \leqq u$ on Δ and so there exists a constant c_v such that $v = c_v u$ on R. So $h = \Pi_R v = c_v \Pi_R u$ as was to be shown. The converse follows similarly since Π_R is one-to-one.

6. In case $0 \leqq \dim PD(R) < \infty$ we have the following *PD*-function analogue corresponding to that for *HD*-functions (cf. **[11]**) and for *PE*-functions (cf. **[2]**):

THEOREM 3. *Δ_p contains exactly m points if and only if* $\dim PD = \dim PBD = m$.

PROOF. First of all if $m=0$, i.e. $\Delta_p = \varnothing$ then any $u \in PD(R)$ vanishes on Δ by Lemma 2 and consequently $u \equiv 0$, i.e. $\dim PD = \dim PBD = 0$. Assume next that there are exactly $m \geqq 1$ points $Z_1, Z_2, \cdots, Z_m \in \Delta_p$.

Take neighborhoods U_i of Z_i such that $\bar{U}_i \cap \bar{U}_j = \varnothing$ $(i \neq j)$ in R^*. Modify (if necessary) each U_i so that ∂U_i is smooth. Choose $h_i \in HBD(U_i)$ such that $h_i|\partial U_i = 0$, $0 \leqq h_i \leqq 1$ on U_i, and $h_i(Z_i) = 1$. As in the proof of Theorem 1 construct functions $u_i \in PD(U_i)$ such that $u_i|\partial U_i = 0$, $u_i(Z_i) = 1$, and $0 \leqq u_i \leqq h_i \leqq 1$ in $\bar{U}_i$. Setting $u_i = 0$ on $R - U_i$ we in turn construct as before $v_i \in PBD(R)$ such that $u_i \leqq v_i \leqq 1$ on R. For a given v_i observe that $v_i(Z_j) = 0$ for $j \neq i$ since the Z_j are regular points for the Dirichlet problem. It follows that the v_i, $i = 1, 2, \cdots, m$, are linearly independent in $PBD(R)$ and so $\dim PD(R) \geqq \dim PBD(R) \geqq m$.

Next let $w \in PD(R)$. Then w has a Riesz decomposition $w = w_1 - w_2$, with $w_i \in PD(R)$, $w_i \geqq 0$ on R. We claim that $w_i(Z_j) < \infty$, $j = 1, \cdots, m$. If not, say $w_i(Z_j) = \infty$; then for $c > 0$, $w_i - cv_j|\Delta \geqq 0$ and so $w_i \geqq cv_j$ on R. But this implies $w_i(Z) = \infty$ for $Z \in R$, a contradiction. Since $w|\Delta - \Delta_p = 0$ it follows that $w = \sum_{i=1}^m (w_1(Z_i) - w_2(Z_i))v_i$ on Δ and hence on R. Therefore $\dim PD = \dim PBD = m$.

Conversely if $\dim PD = \dim PBD = m$ then Δ_p cannot contain more than m points. For if there exist at least $m+1$ points $Z_1, Z_2, \cdots, Z_{m+1} \in \Delta_p$ then as in the first part of the proof construct $m+1$ linearly independent functions $v_1, v_2, \cdots, v_{m+1} \in PBD$, thereby contradicting $\dim PBD = m$. Hence Δ_p has n points $0 \leqq n \leqq m$. As earlier in the proof there exist n functions $v_i \in PBD(R)$ such that any $w \in PD(R)$ is a linear combination of these v_i. We conclude Δ_p has precisely m points; and this completes the proof.

Added in proof. Results similar to those in this paper have been obtained by J. L. Schiff (*A note on the space of Dirichlet-finite solutions of $\Delta u = Pu$ on a Riemann surface*, Hiroshima Math. J. **2** (1972) (to appear).)

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