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A NEW TOPOLOGY ON B^*-ALGEBRAS ARISING FROM THE ARENS PRODUCTS

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ABSTRACT. A locally convex topology μ is defined on a Banach algebra A. This topology arises naturally from considerations of the Arens products on the second conjugate space A^{**} of A. The main result states that if A is a B^*-algebra on which the mapping $(a, b)\rightarrow ab$ is μ-continuous for $\|a\|\leqq 1$, then the completion of A with respect to the uniformity generated by μ is linearly isomorphic to A^{**}. An example is included which shows that this continuity condition does not hold in general as announced by P. C. Shields.

Properties of the Arens products on the second conjugate space of a Banach algebra have been widely investigated. In attempting to extend the Arens products to locally convex completions of certain algebras, a particular locally convex topology μ came to our attention.

In §1 we define this topology on a Banach algebra A and investigate some of its properties when extended to A^{**}. In §2 we show that if A is a certain type of B^*-algebra, then the completion of A with respect to μ is isomorphic to the second conjugate space of A under the extended μ topology: Thus, in particular, this completion is an algebra under either Arens product. In §3 we discuss a topology defined on W^*-algebras similar to μ which was announced by P. C. Shields [7] and give a counterexample to one of his assertions.

1. **The μ topology.** Let A be a Banach algebra and A^* denote its norm dual. For $f\in A^*$ and $a\in A$ define two bounded linear functionals, $f\cdot a$ and $a\cdot f$, and a seminorm p_f on A in the following way: (i) $f\cdot a(x)=f(ax)$; (ii) $a\cdot f(x)=f(xa)$; and (iii) $p_f(x)=\max\{\|f\cdot x\|, \|x\cdot f\|\}$. The μ topology on A is the locally convex topology determined by the set of seminorms $\{p_f : f\in A^*\}$.

PROPOSITION 1. *If A has a bounded approximate identity $\{e_\alpha\}$, then the following hold*: (1) *the μ topology on A is Hausdorff*; (2) *the set of μ-continuous linear functionals on A, denoted by $(A, \mu)^*$, coincides with A^**;

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(3) *for each* $a \in A$ *the mappings* $x \to xa$ *and* $x \to ax$ *on* A *are* μ*-continuous*; *and* (4) *if*, *moreover*, A *has an isometric involution*, *then it is also* μ*-continuous.*

PROOF. (1) If $a \neq 0$, there is $f \in A^*$ such that $f(a) > 0$ and there exists e in the bounded approximate identity such that $\max\{\|a - ae\|, \|a - ea\|\} < (a)/2\|f\|$. Therefore, $\max\{\|f \cdot a\|, \|a \cdot f\|\} \geqq f(a)/2 > 0$ and so μ is Hausdorff. (2) follows from the Mackey-Arens theorem and the following inequality: $|f(x)| \leqq \min\{\sup|f(xe_\alpha)|, \sup|f(e_\alpha x)|\} \leqq \max\{\|f \cdot x\|, \|x \cdot f\|\} \leqq \|f\| \; \|x\|$. (3) is easily verified, and (4) follows readily by defining for $f \in A^*$ a bounded linear functional f^* on A by $f^*(x) = \overline{f(x^*)}$, whence $\|f \cdot x^*\| = \|x \cdot f^*\|$. This completes the proof.

The μ topology can be extended to A^{**} where its relationship to the $\sigma(A^{**}, A^*)$ and $\tau(A^{**}, A^*)$ topologies is connected with the regularity of the Arens products. For $F \in A^{**}$ and $f \in A^*$ define two linear functionals $F \cdot f$ and $F{:}f$ on A and a seminorm P_f on A^{**} as follows: (i) $F \cdot f(x) = F(f \cdot x)$; (ii) $F{:}f(x) = F(x \cdot f)$; and (iii) $P_f(F) = \max\{\|F \cdot f\|, \|F{:}f\|\}$. The μ topology on A^{**} is the locally convex topology determined by the set of seminorms $\{P_f : f \in A^*\}$.

Arens [1] has defined two products on A^{**} under which it is a Banach algebra with respect to the usual norm: For $F, G \in A^{**}$ define $F \cdot G$ by $F \cdot G(f) = F(G \cdot f)$ and $F{:}G$ by $F{:}G(f) = F(G{:}f)$. If $F \cdot G = G{:}F$ for all $F, G \in A^{**}$ the Arens products on A^{**} are said to be *regular*.

PROPOSITION 2. *The following are true*: (1) *if there exists a left identity* I *for either Arens product*, *then* μ *on* A^{**} *is finer than* $\sigma(A^{**}, A^*)$; *and* (2) μ *on* A^{**} *is coarser than* $\tau(A^{**}, A^*)$ *if and only if the Arens products are regular.*

PROOF. (1) follows since $|F(f)| \leqq \|I\| \max\{\|F \cdot f\|, \|F{:}f\|\}$. To prove (2) first assume that μ is coarser than $\tau(A^{**}, A^*)$. Choose $F, G \in A^{**}$ such that $\|F\| \leqq 1$. Since the unit ball of A is convex and $\sigma(A^{**}, A^*)$-dense in the unit ball of A^{**}, it is also $\tau(A^{**}, A^*)$-dense. Choose a net $\{a_\alpha\}$ in the unit ball of A which converges to F in the $\tau(A^{**}, A^*)$ topology. Then $\lim\|F \cdot f - a_\alpha \cdot f\| = 0$ so that $G \cdot F(f) = \lim G(a_\alpha \cdot f) = \lim(G{:}f)(a_\alpha) = F{:}G(f)$.

Conversely, suppose that the Arens products are regular. In A^{**} let $\{F_\alpha\}$ be a net converging to 0 in the $\tau(A^{**}, A^*)$ topology and let $f \in A^*$. The net $\{F_\alpha\}$ converges to 0 uniformly on every absolutely convex $\sigma(A^*, A^{**})$-compact subset of A^* and, in particular, on the sets $S \cdot f = \{a \cdot f : \|a\| \leqq 1, a \in A\}$ and $f \cdot S = \{f \cdot a : \|a\| \leqq 1, a \in A\}$ by [5, Theorem 2.1]. Thus

$$\lim(\max\{\|F_\alpha{:}f\|, \|F_\alpha \cdot f\|\}) = 0$$

and the proof is completed.

COROLLARY 1. *If A is a Banach algebra with bounded approximate identity $\{e_\alpha\}$ such that* $\lim \|f \cdot e_\alpha - f\| = 0$ *for all* $f \in A^*$, *then* μ *on* A^{**} *is finer than* $\sigma(A^{**}, A^*)$.

PROOF. By Alaoglu's theorem choose $I \in A^{**}$ to be a $\sigma(A^{**}, A^*)$-cluster point of the bounded approximate identity. Then $F(f) = \lim F(f \cdot e_\alpha) = I \cdot F(f)$ for each $f \in A^*$, $F \in A^{**}$.

COROLLARY 2. *If A has a bounded approximate identity and the Arens products are regular, the following hold*:

(1) *the set of μ-continuous linear functionals on A^{**} coincides with A^*; and*

(2) *the mappings $F \to F \cdot G$ and $F \to G \cdot F$ on A^{**} are μ-continuous.*

PROOF. Choose I as in Corollary 1. Then $\lim f(ae_\alpha) = f(a) = \lim f(e_\alpha a)$ so that $I \cdot f = f = I{:}f$ for each $f \in A^*$. By regularity I is an identity for both Arens products. (2) is easily verified.

We note here that if A is a B^*-algebra, then the hypotheses of Corollary 2 are satisfied [**4**, p. 15]; [**2**, Theorem 7.1].

2. **The continuity condition on B^*-algebras.** In this section we prove the main result:

THEOREM. *If A is a B^*-algebra such that the mapping $(a, b) \to ab$ on A is μ-continuous for $\|a\| \leq 1$, then A^{**} is linearly isomorphic to $(A, \mu)^\wedge$, the completion of A with respect to the uniform structure generated by μ.*

We freely use the following: Grothendieck's completeness theorem to identify $(A, \mu)^\wedge$ with a certain set of linear functionals on A^*; the fact that A^{**} is a W^*-algebra under the Arens product $(F, G) \to F \cdot G$ [**2**]; and Sakai's representation theorem for W^*-algebras [**6**].

LEMMA 1. *Let P denote the set of positive linear functionals on A. Suppose that $\{F_\alpha\}$ is a net of hermitian elements in the unit ball of A^{**} such that* $\lim F_\alpha \cdot F_\alpha(\theta) = 0$ *for some $\theta \in P$. If Q denotes the support of θ in A^{**}, then $\{Q \cdot F_\alpha \cdot Q\}$ converges to zero in the μ topology.*

PROOF. The elements of P are exactly the normal positive linear functionals on A^{**} [**8**]. We shall first show that $\lim Q \cdot F_\alpha \cdot Q \cdot Q \cdot F_\alpha \cdot Q(\phi) = 0$ for all $\phi \in P$. If A^{**} is represented as a ring of operators on the Hilbert space H, then for any $\phi \in P$ there exists a sequence $\{x_i\}$ in H such that $\sum \|x_i\|^2 < \infty$ and $\phi = \sum [x_i, x_i]$, where $[x_i, x_i](F) = ((Fx_i, x_i))$ with $((\ ,\ ))$ denoting inner product in H [**3**, p. 51].

For $\varepsilon > 0$ choose N such that $\sum_{i=N+1}^{\infty} \|x_i\|^2 < \varepsilon/2$. Since

$$\lim F_\alpha \cdot Q \cdot Q \cdot F_\alpha(\theta) = \lim \|Q\| F_\alpha \cdot F_\alpha(\theta) = 0$$

[**4**, p. 23], the net $\{Q \cdot F_\alpha \cdot Q\}$ converges to 0 in the strong operator topology [**3**, p. 58]. Choosing α sufficiently large, we have that $\|Q \cdot F_\alpha \cdot Qx_i\| < \varepsilon/2N$ for $i=1, \cdots, N$. Thus $\sum \|Q \cdot F_\alpha \cdot Qx_i\|^2 < \varepsilon$ for sufficiently large α.

Since P spans A^* [**8**], for $f \in A^*$ there exists a finite number of $\phi_i \in P$ such that $|F{:}f(x)| \leqq \sum |F \cdot x(\phi_i)|$ for any $F \in A^{**}$. By the Cauchy-Schwarz inequality we then have $\|F{:}f\| \leqq \sum \|\phi_i\|^{1/2} F \cdot F^*(\phi_i)^{1/2}$. A similar inequality can be obtained for $\|F \cdot f\|$. The proof is completed by taking $F = Q \cdot F_\alpha \cdot Q$.

LEMMA 2. *Let A be a B^*-algebra such that the mapping $(a, b) \to ab$ is μ-continuous for $\|a\| \leqq 1$. Let S' denote the unit ball of A^{**}. Then for each μ-neighborhood U' of zero in A^{**} there exists a μ-neighborhood V of zero in A and a μ-neighborhood W' of zero in A^{**} such that $V \cdot (W' \cap S') \subset U'$.*

PROOF. The neighborhood U' contains a set of the form

$$\{F \in A^{**} : \|F \cdot f_n\| \leqq 1, \|F{:}f_n\| \leqq 1, n = 1, \cdots, N\}.$$

Let $U = \frac{1}{2}(U' \cap A)$. Since involution on A is μ-continuous, the mapping $(a, b) \to ab$ on A for $\|b\| \leqq 1$ is μ-continuous; hence we can choose μ-neighborhoods V and W of 0 in A such that $V \cdot (W \cap S) \subset U$, where S denotes the unit ball of A. Since A has a bounded approximate identity, from Proposition 1 we have $A^* = (A, \mu)^*$ and it follows that $V \cdot (W \cap S)^{\circ\circ} \subset U^{\circ\circ}$, where the bipolars are taken in A^{**}.

Let $W' = W^{\circ\circ}$ in A^{**}. Noting that $(W \cap S)^\circ$ in A^* is the $\sigma(A^*, A)$-closed absolutely convex hull of $W^\circ \cup S^\circ$ and that W° and S° are $\sigma(A^*, A)$-compact by Alaoglu's theorem, we have that $(W \cap S)^\circ \subset W^\circ + S^\circ$. Thus, $V \cdot (W' \cap S') \subset V \cdot 2(W \cap S)^{\circ\circ} \subset 2U^{\circ\circ} \subset U'$.

It remains to show that W' is a μ-neighborhood of zero in A^{**}. If W contains a set of the form $\{a \in A : \|g_n \cdot a\| \leqq 1, \|a \cdot g_n\| \leqq 1, n = 1, \cdots, N\}$, then W' contains the set $\{F \in A^{**} : \|F \cdot g_n\| \leqq \frac{1}{2}, \|F{:}g_n\| \leqq \frac{1}{2}\}$. For each such F there exists a net $\{a_\alpha\}$ in A such that F is its $\sigma(A^{**}, A^*)$-limit. Since the Arens products are regular in A^{**}, for each x in A we have that $x \cdot F$ and $x{:}F$ are the $\sigma(A^{**}, A^*)$-limits of the nets $\{xa_\alpha\}$ and $\{a_\alpha x\}$ respectively [**1**]. Therefore, $\max\{\|a_\alpha \cdot g_n\|, \|g_n \cdot a_\alpha\|\} \leqq 1$ for sufficiently large α; thus $F \in W' = W^{\circ\circ}$ which is $\sigma(A^{**}, A^*)$-closed.

LEMMA 3. *If A is a B^*-algebra and if a net $\{a_\alpha\}$ in A converges to $T \in (A, \mu)^\wedge$ in the completion topology, then*

$$|T(f)| \leqq \sup\{|T(f \cdot x)| : \|x\| \leqq 1, x \in A\} = \lim \|a_\alpha \cdot f\| < \infty$$

for each $f \in A^*$.

PROOF. Let $\{x_n\}$ be a sequence in the unit ball of A which converges to 0 in norm. Then the sequence $\{f \cdot x_n\}$ is contained in the polar of the

μ-neighborhood $U=\{x \in A: \|f \cdot x\| \leqq 1, \|x \cdot f\| \leqq 1\}$. Since this sequence certainly converges to 0 in the $\sigma(A^*, A)$ topology, $\lim T(f \cdot x_n)=0$. Thus $T \cdot f$ defined on A by $T \cdot f(x)=T(f \cdot x)$ is norm-continuous.

For $\varepsilon>0$ and U as above $\varepsilon U^{\circ\circ} \cap (A, \mu)\hat{}$ in $A^{*\prime}$, the set of linear functionals on A^*, is a neighborhood of 0 in the completion topology. Thus $T-a_\alpha$ is in this neighborhood for sufficiently large α. Since $f \cdot x \in U^\circ$ for each $x \in A$ with $\|x\| \leqq 1$, it follows that $\|T \cdot f\| = \lim \|a_\alpha \cdot f\|$.

If I is a $\sigma(A^{**}, A^*)$-cluster point of a bounded approximate identity $\{e_\alpha\}$ for A, then $I{:}f=f$ for all $f \in A^*$ (see Corollary 2), and thus by regularity we have that f is a $\sigma(A^*, A^{**})$-cluster point of the net $\{f \cdot e_\alpha\}$ **[1]**. Hence $|T(f)| \leqq \sup |T(f \cdot e_\alpha)| \leqq \|T \cdot f\|$. This completes the lemma.

Proof of the Theorem. For $F \in A^{**}$ choose a net in A for which F is its $\tau(A^{**}, A^*)$-limit. Since the Arens products are regular, F is also the μ-limit of this net by Proposition 2. Since the net in A is μ-Cauchy, it follows that $F \in (A, \mu)\hat{}$.

To show that $(A, \mu)\hat{}$ is contained in A^{**}, it suffices to show that if $\{\theta_n\}$ is a sequence of positive linear functionals in A^* such that $\|\theta_n\| \leqq 1/2^n$ then $\lim T(\theta_n)=0$ for $T \in (A, \mu)\hat{}$ **[8]**.

Let $\theta=\sum \theta_n$; then θ is a normal positive linear functional on A^{**} and $0 \leqq F^* \cdot F(\theta_n) \leqq F^* \cdot F(\theta)$ for all F in A^{**}. Choose a sequence $\{G_n\}$ in A^{**} such that $0 \leqq G_n \leqq 1$ and $F(\theta_n)=G_n \cdot F \cdot G_n(\theta)$ for all F in A^{**} [3, p. 63]; then if Q denotes the support of θ, we have that $0 \leqq Q \cdot G_n \cdot Q \cdot Q \cdot G_n \cdot Q(\theta)= Q(\theta_n) \leqq \|\theta_n\|$. Consequently, the sequence $\{Q \cdot G_n \cdot Q\}$ converges to 0 in the μ topology by Lemma 1.

Now let $\varepsilon>0$ and consider the μ-neighborhood $U'=\{F \in A^{**}: \|F \cdot \theta\| \leqq \varepsilon/2, \|F{:}\theta\| \leqq \varepsilon/2\}$ in A^{**}. By Lemma 2 there exists a μ-neighborhood V of 0 in A and a μ-neighborhood W' of 0 in A^{**} such that $V \cdot (W' \cap S') \subset U'$, where S' denotes the unit ball of A^{**}.

Let $\{a_\alpha\}$ be a μ-Cauchy net in A converging to $T \in (A, \mu)\hat{}$. Fix a in this net such that $a-a_\alpha \in V$ for sufficiently large α. Choose N such that $\|\theta_n\| < \varepsilon/2(\|a\|+1)$ for $n>N$, and M such that $Q \cdot G_n \cdot Q \in W'$ for $n>M$. Let $F=(a_\alpha-a) \cdot Q \cdot G_n \cdot Q$ for $n>N+M$ and α sufficiently large so that $F \in V \cdot (W' \cap S') \subset U'$. Then $\|(Q \cdot G_n \cdot Q){:}(F \cdot \theta)\| \leqq \varepsilon/2$. From the relations on Q and G_n and [3, p. 59] we have for $x \in A$ that

$$\begin{aligned}(Q \cdot G_n \cdot Q){:}(F \cdot \theta)(x) &= Q \cdot x \cdot (a_\alpha - a) \cdot Q(\theta_n) \\ &= [x \cdot (a_\alpha - a)](\theta_n) = (a_\alpha - a) \cdot \theta_n(x).\end{aligned}$$

Therefore, $\|a_\alpha \cdot \theta_n\| < \varepsilon$ for $n>N+M$ and sufficiently large α. Thus by Lemma 3 we have that $|T(\theta_n)|$ is bounded by the limit with α of the net $\{\|a_\alpha \cdot \theta_n\|\}$, which is at most ε for n large.

Therefore, A^{**} and (A, μ) are equal as sets. That the μ topology and the completion topology are the same is essentially proved in Lemma 2.

3. **The continuity condition on W^*-algebras.** A topology on W^*-algebras similar to the extended μ topology was announced by P. C. Shields [7] for which he claimed that the mapping $(a, b)\to ab$ (for $\|a\|\leqq 1$) on an arbitrary W^*-algebra is continuous. In this section we give a counterexample to his assertion.

Let M be a W^*-algebra with predual F. Conventionally we consider F as a set of linear functionals on M and so use the notation established previously. Shields' topology is determined by the seminorms $x\to \max\{\|f\cdot x\|, \|x\cdot f\|\}$ for all $f\in F$. We shall also refer to this topology as μ. If M is commutative, the mapping $(a, b)\to ab$ for $\|a\|\leqq 1$ is μ-continuous: It follows directly from the identity $ab-a_\alpha b_\alpha=(a-a_\alpha)b+a_\alpha(b-b_\alpha)$. The next proposition establishes the means by which we construct a counterexample for the general case.

Proposition 3. *Suppose there exist $\varepsilon>0$ and a positive linear functional θ in F which satisfy the following property: For every $\delta>0$ and for every positive linear functional ϕ in F there exist equivalent projections p and q in M such that $\theta(p)>\varepsilon$ and $\phi(q)<\delta$. Then the mapping $(a, b)\to ab$ on M is not μ-continuous for $\|a\|\leqq 1$.*

Proof. Let U be the μ-neighborhood $\{x\in M: \|\theta\cdot x\|\leqq\varepsilon, \|x\cdot\theta\|\leqq\varepsilon\}$; let $V=\{x\in M: \|\psi_k\cdot x\|\leqq\sigma, \|x\cdot\psi_k\|\leqq\sigma, k=1,\cdots,K\}$ and $W=\{x\in M: \|\phi_n\cdot x\|\leqq\eta, \|x\cdot\phi_n\|\leqq\eta, n=1,\cdots,N\}$ be any two basic μ-neighborhoods of 0. Choose $\lambda>0$ such that $\lambda\leqq\min\{1, \sigma/(1+\sum\|\psi_k\|)\}$. Let $\phi=\sum\phi_n$ and $\delta=[\lambda\eta/(1+\sum\|\phi_n\|^{1/2})]^2$. Then by hypothesis for ϕ and δ there exist projections $p, q\in M$ and $v\in M$ such that $vv^*=p$, $v^*v=q$, $\theta(p)>\varepsilon$ and $\phi_n(q)\leqq\phi(q)<\delta$ for all n.

Since $\|v\|=1$ so that $\max\{\|\lambda v\cdot\psi_k\|, \|\psi_k\cdot\lambda v\|\}\leqq\sigma$ for all k, we see that $\lambda v\in V\cap S$, where S denotes the unit ball of M. Furthermore, as a consequence of the Cauchy-Schwarz inequality we have that

$$\max\{\|(1/\lambda)q\cdot\phi_n\|, \|\phi_n\cdot(1/\lambda)q\|\}\leqq(1/\lambda)\|\phi_n\|^{1/2}\phi_n(q)^{1/2}\leqq\eta,$$

for all n; hence $(1/\lambda)q\in W$ and $vq\in(V\cap S)\cdot W$.

Noting that $qv^*=v^*$, we have $\|\theta\cdot vq\|\geqq\theta(vqv^*)=\theta(p)>\varepsilon$. Thus the mapping $(a, b)\to ab$ is not μ-continuous for $\|a\|\leqq 1$ at the point $a=0=b$.

Corollary. *If M is a W^*-algebra containing an infinite number of nonzero equivalent projections, then the mapping $(a, b)\to ab$ on M is not μ-continuous for $\|a\|\leqq 1$.*

Proof. Let $\{p_n\}$ be an infinite set of nonzero equivalent orthogonal projections. Since F is spanned by the normal positive linear functionals on M, there is one such functional θ for which $\theta(p_1)>0$. For any positive linear functional $\phi \in F$ and $\delta>0$ there exists N such that $\phi(p_N)<\delta$. If this were not true, then by choosing n such that $n\delta>\|\phi\|$ we have $n\delta\leqq \phi(\sum_{i=1}^n p_i)\leqq\|\phi\|$ since ϕ is linear and the sum of these projections is a projection, a contradiction. The conclusion now follows from the proposition with $\varepsilon=\theta(p_1)$.

Thus, if H is an infinite dimensional Hilbert space, the set of all bounded linear functionals on H clearly satisfies the hypothesis of the above Corollary.

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