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THE CLOSED IMAGE OF A METRIZABLE SPACE IS M_1

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ABSTRACT. J. Ceder introduced the notions of M_1 space (a regular space with σ-closure preserving base) and stratifiable space as natural generalizations of Nagata and Smirnov's conditions for the metrizability of a regular space. Even though a topological space Y which is the image of a metrizable space under a closed, continuous mapping need not be metrizable, we show as our main result that Y will have a σ-closure preserving base. It follows that one cannot obtain an example of a stratifiable space which is not M_1 by constructing a quotient space from an upper semicontinuous decomposition of a metric space. In the course of establishing our major result, we obtain conditions under which the image of certain collections of sets under a closed, continuous mapping will be closure preserving.

1. **Introduction.** It is well known ([5] [6]) that the image of a metrizable space under a closed, continuous mapping need not be metrizable even though such a space will have considerable structure. Indeed, the closed, continuous images of metric spaces (now called Lašnev spaces) were characterized internally by Lašnev [4]. Although the Nagata-Smirnov metrization theorem makes it clear that a nonmetrizable Lašnev space will not have a σ-locally finite base; nevertheless, as we show in Theorem 6, every Lašnev space has a σ-closure preserving base.

Regular spaces with σ-closure preserving bases were studied by J. Ceder [2] who called them M_1 spaces. In the same paper, Ceder also defined the class of stratifiable spaces (which he called M_3 spaces) and showed that metrizable spaces are M_1 spaces and M_1 spaces are stratifiable. However, Ceder's question of whether a stratifiable space is also an M_1 space remains open. In searching for an example which is stratifiable but not M_1, it is natural to investigate the effect of applying a mapping f which preserves stratifiability to a suitable M_1 space X in the hope that $f(X)$ would fail to be M_1. Theorem 6 of course shows that one cannot obtain a stratifiable space (the closed, continuous image of a stratifiable space is

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stratifiable [1, Theorem 3.1, p. 5]) which is not an M_1 space as the closed, continuous image of a metrizable space.

The major purpose of this paper is to obtain Theorem 6. In the course of proving Theorem 6, we obtain Lemma 3 and Lemma 4 which give conditions under which the closed image of certain collections of sets will be closure preserving.

2. **Definitions and notation.** Recall that a regular topological space X is an M_1 space if there is a sequence $\{\mathscr{B}_i\}$ of open collections each of which is closure preserving and for which $\mathscr{B}=\bigcup\{\mathscr{B}_i:i=1,2,\cdots\}$ is a base for X.

Let $\mathscr{U}=\{U_\alpha:\alpha\in A\}$ be a collection of subsets of the set X. For any $B\in P(A)$, $P(A)$ being the collection of all subsets of A, define $V(B)$ by $V(B)=\bigcup\{U_\alpha:\alpha\in B\}$. Suppose f is a function from the set X into the set Y. For $y\in Y$, we sometimes denote $f^{-1}(y)$ by $\tilde{y}$. Also if $T\subseteq X$, the saturated part of T, T_s, is defined by

$$T_s=\bigcup\{f^{-1}(y):f^{-1}(y)\subseteq T\}.$$

For the collection $\mathscr{U}=\{U_\alpha:\alpha\in A\}$ of subsets of X, we can consider the collection $\mathscr{W}(\mathscr{U})$ of all saturated parts of unions of sets in $\mathscr{U}$ i.e.

$$\mathscr{W}(\mathscr{U})=\{W(B)=[V(B)]_s:B\in P(A)\}.$$

Also set $\mathscr{Z}(\mathscr{U})=\{f(W):W\in\mathscr{W}(\mathscr{U})\}$. The notation $V(B)$, $\mathscr{W}(\mathscr{U})$, $\mathscr{Z}(\mathscr{U})$, $\tilde{y}$ will be used throughout this paper.

If Y is a topological space, $Y_0\subseteq Y$, and $\mathscr{B}$ a collection of open subsets of Y, we say that $\mathscr{B}$ is a *base for the points of* Y_0 provided that for any $y\in Y_0$ and open set O containing y, there is a $B\in\mathscr{B}$ with $y\in B\subseteq O$.

All mappings in this paper are at least continuous.

3. **Preliminary results.**

LEMMA 1. *Let M be a metrizable space and let f be a closed mapping from X onto Y. Let $Y_0=\{y:f^{-1}(y)$ is compact$\}$. Then there is a base $\mathscr{B}$ for the points of Y_0 which is σ-locally finite.*

PROOF. Let ρ be a metric on M compatible with the topology of M. Let $\{\mathscr{U}_i\}$ be a sequence of open covers of M with mesh $\mathscr{U}_i$ going to 0. For each i let $\mathscr{U}_i=\{U_{i,\alpha}:\alpha\in A_i\}$ and for $j\geqq 1$, let

$$\mathscr{V}_{i,j}=\{U_{i,\alpha_1}\cup U_{i,\alpha_2}\cup\cdots\cup U_{i,\alpha_j}:\alpha_k\in A_i\text{ and all }\alpha_k\text{ distinct}\}.$$

Thus $\mathscr{V}_{i,j}$ consists of all exactly j-fold unions of elements of $\mathscr{U}_i$. Consider $\mathscr{C}_{i,j}=\{f(V_s):V\in\mathscr{V}_{i,j}\}$ where V_s is the saturated part of V. Clearly since f is a closed mapping, $\mathscr{C}_{i,j}$ is a collection of open subsets of Y.

Claim. For any $y \in Y_0$ and $O(y)$ an open neighborhood of y, there are an i and a j so that $y \in St(y, \mathscr{C}_{i,j}) \subseteq O(y)$.

To establish the claim, note that since $f^{-1}(y)$ is compact,

$$\rho(f^{-1}(y), X - f^{-1}(O(y))) = \varepsilon > 0.$$

Choose i so that mesh $U_i < \varepsilon$. Also choose a minimal j so that $f^{-1}(y)$ is covered by j distinct sets of $\mathscr{U}_i$. Suppose that $U_{i,\alpha_1}, U_{i,\alpha_2}, \cdots, U_{i,\alpha_j}$ are any j distinct sets of $\mathscr{U}_i$ which cover $f^{-1}(y)$. Then since j is minimal, each $U_{i,\alpha}$ must hit $f^{-1}(y)$. Thus

$$f^{-1}(y) \subseteq U_{i,\alpha_1} \cup U_{i,\alpha_2} \cup \cdots \cup U_{i,\alpha_j} \subseteq f^{-1}(O(y)).$$

It follows that there is a $C \in \mathscr{C}_{i,j}$ containing y. Moreover if C' is any element of $\mathscr{C}_{i,j}$ containing y, then $C' \subseteq O(y)$. Hence $y \in St(y, \mathscr{C}_{i,j}) \subseteq O(y)$.

For each i, j let $R_{i,j} = \bigcup \{C : C \in \mathscr{C}_{i,j}\}$. Since Y is a normal space whose open sets are F_σ's, for each i, j we have countably many open sets $\{S_{i,j,k} : k = 1, 2, \cdots\}$ with

$$\bigcup \{S_{i,j,k} : k = 1, 2, \cdots\} = \bigcup \{\mathrm{Cl}(S_{i,j,k}) : k = 1, 2, \cdots\} = R_{i,j}.$$

Using paracompactness of Y, each $\mathscr{C}_{i,j}$ restricted to $S_{i,j,k}$ has an open refinement $\mathscr{B}_{i,j,k}$ which covers $S_{i,j,k}$ and is locally finite in Y. Setting $\mathscr{B} = \bigcup \{\mathscr{B}_{i,j,k} : i, j, k = 1, 2, \cdots\}$, we have a σ-locally finite base for the points of Y_0 as desired.

Note that with the help of the Nagata-Smirnov metrization theorem and Lemma 1, we obtain the well-known results of Morita-Hanai [**5**, Theorem 1, p. 11] and Stone [**6**, Theorem 1, p. 691] that a perfect image of a metric space is metrizable.

Lemma 2 often appears implicitly in the literature concerning closed mappings; consequently its proof will be omitted.

LEMMA 2. *Let f be a closed mapping from the T_1 space X into the space Y. Let $\{y_i\}$ be a sequence of distinct points of Y converging to y. Then any subsequence $\{x_{i_j}\}$ with $x_{i_j} \in \tilde{y}_{i_j}$ has a cluster point.*

It will be useful in what follows to have conditions under which the closed image of a collection of sets will be closure preserving. Lemmas 3 and 4 provide such conditions.

LEMMA 3. *Let f be a closed mapping from the T_1 space X into the Fréchet and Hausdorff space Y. Let $\mathscr{U}$ be an hereditarily closure preserving collection in X. Then $\mathscr{Z}(\mathscr{U})$ is closure preserving.*

PROOF. Let $\mathscr{Z}' \subseteq \mathscr{Z}(\mathscr{U})$. For suitable $\mathscr{Q} \subseteq P(A)$ we have $\mathscr{Z}' = \{f([V(B)]_s) : B \in \mathscr{Q}\}$. Set $S = \bigcup \{Z : Z \in \mathscr{Z}'\}$ and let $y \in \mathrm{Cl}(S) - S$. Since Y is a Hausdorff and Fréchet space, there is a sequence $\{y_i\}$ of distinct points of S converging to y. Each $\tilde{y}_i$ is contained in a $V(B_i)$ for suitable $B_i \in \mathcal{Q}$. For each $i=1, 2, \cdots$, define C_i by

$$C_i = \{\alpha : \alpha \in B_i \text{ and } U_\alpha \cap \tilde{y}_i \neq \varnothing\}.$$

Clearly $\tilde{y}_i \subseteq V(C_i)$. Moreover we will show that at most finitely many C_i's are infinite. Clearly if this is not the case, then there is a sequence of distinct α_{i_j}'s with $\alpha_{i_j} \in C_{i_j}$. Choosing points $x_{i_j} \in U_{\alpha_j} \cap \tilde{y}_{i_j}$, we have a discrete sequence in violation of Lemma 2.

Without loss of generality, assume that each C_i is finite. Indeed we have that $C = \bigcup \{C_i : i=1, 2, \cdots\}$ is finite; if not again choose a discrete sequence of points from $\tilde{y}_{i_j} \cap U_{\alpha_{i_j}}$ for suitable distinct α_{i_j}'s and y_{i_j}'s in violation of Lemma 2.

Thus the range of the mapping $\tilde{y}_i \to C_i$ lies in the finite set $P(C)$. Consequently there is C_{i_0} and sequence C_{i_j} with $C_{i_0} = C_{i_j}$ for $j=1, 2, \cdots$. Then for each j, $\tilde{y}_{i_j} \subseteq V(C_{i_j}) = V(C_{i_0}) \subseteq V(B_{i_0})$. Thus $y \in \mathrm{Cl}(f(V(B_{i_0})_s))$ from which it follows that $\mathcal{L}'$ is closure preserving.

We note that D. Lutzer has generalized Lemma 3 by deleting the assumption that X is T_1 and Y is Hausdorff.

By requiring the mapping f in Lemma 3 to be perfect and the collection $\mathcal{U}$ to be locally finite, we can obtain the conclusion of Lemma 3 without restriction on Y. More precisely, we have

LEMMA 4. *Let $f: X \to Y$ be a perfect mapping (i.e. a closed mapping with compact fibers) and let $\mathcal{U}$ be a locally finite collection of subsets of X. Then $\mathcal{L}(\mathcal{U})$ is closure preserving.*

LEMMA 5. *Let $f: M \to Y$ be a closed mapping from the metrizable space M onto the space Y. Then each point y in Y has a closure preserving local base $\mathcal{D}_y$ of open neighborhoods.*

PROOF. Let y in Y be fixed. By choosing a metric ρ on M compatible with the topology of M, we can obtain a sequence $\{\mathcal{U}_j\}$ of locally finite open collections with mesh $\mathcal{U}_j < 1/j$ so that every element of each $\mathcal{U}_j$ hits $\tilde{y}$. For each j let $\mathcal{U}_j = \{U_{j,\alpha} : \alpha \in A_j\}$, $\mathcal{V}_j = \{V(B) : B \in P(A_j)\}$ and

$$\mathcal{W} = \left\{\bigcup_{j=1}^{\infty} V_j : V_j \in \mathcal{V}_j \text{ and } \bigcup_{j=1}^{\infty} V_j \supseteq \tilde{y}\right\}.$$

We assert that $\mathcal{D}_y = \{f(W_s) : W \in \mathcal{W}\}$ is a closure preserving system of open neighborhoods of y which forms a local base at y.

Clearly $\mathcal{D}_y$ is a system of open neighborhoods of Y. Moreover it is easy to see that since f is a closed mapping, $\mathcal{D}_y$ is a local base at y. It remains to show that $\mathcal{D}_y$ is closure preserving. To do so, let $\mathcal{D}' \subseteq \mathcal{D}_y$ and let

$$S = \bigcup \{D : D \in \mathcal{D}'\} = \bigcup \{f(W_s) : W \in \mathcal{W}'\}$$

for a suitable $\mathscr{W}'\subseteq\mathscr{W}$. Let $z\in \mathrm{Cl}(S)-S$. Since Y is a Fréchet space, there is a sequence $\{y_i\}$ of distinct points of S converging to z. Choose $W_i\in\mathscr{W}'$ so that $\tilde{y}_i\subseteq W_i$. For each i, we have $W_i=\bigcup\{V_{i,j}:j=1,2,\cdots\}$ where $V_{i,j}\in\mathscr{V}_j$.

Claim. There is an m so that for all but finitely many i's, if $j>m$, then $V_{i,j}\cap\tilde{y}_i=\varnothing$.

To establish the claim suppose the contrary. Then there is a sequence $\{\tilde{y}_{i_k}\}$ so that $\{\tilde{y}_{i_k}\}$ hits $\bigcup\{V_{i_k,j}:j>k\}$ for each $k=1,2,\cdots$. Choose sequences $\{p_{i_k}\}$ and $\{q_{i_k}\}$ with $p_{i_k}\in\tilde{y}_{i_k}$, $q_{i_k}\in\tilde{y}$ and $\rho(p_{i_k},q_{i_k})<1/k$. According to Lemma 2, $\{p_{i_k}\}$ has a cluster point p. Hence some subsequence of $\{p_{i_k}\}$ converges to p, the image of this subsequence converges to z and consequently $f(p)=z$. But since $\rho(p_{i_k},q_{i_k})\to 0$, a subsequence of $\{q_{i_k}\}$ converges to p and $f(p)=y$. But $y\neq z$ and therefore the claim is established.

Having established the claim, without loss of generality we can assume there is an m so that, for any i, $\tilde{y}_i\subseteq\bigcup\{V_{i,j}:1\leqq j\leqq m\}=O_i$. Each $\tilde{y}_i$, by virtue of being contained in O_i, lies in a union of elements from the locally finite collection $\mathscr{U}_1\cup\mathscr{U}_2\cup\cdots\cup\mathscr{U}_m$. By Lemma 3, $\{f([O_i]_s):i=1,2,\cdots\}$ is closure preserving and thus for some i_0, $z\in\mathrm{Cl}(f([O_{i_0}]_s))$. But $f([O_{i_0}]_s)$ is contained in some $D\in\mathscr{D}'$ and $z\in\bigcup\{\mathrm{Cl}(D):D\in\mathscr{D}'\}$. Therefore $\mathscr{D}_y$ is closure preserving as desired.

4. **Main result.** In [2, Theorem 7.6, p. 117], Ceder established that the quotient space obtained by identifying a closed set in a metric space to a point is an M_1 space. Theorem 6 provides a considerably more general result.

THEOREM 6. *Let f be a closed mapping from the metric space M onto the space Y. Then Y is an M_1 space.*

PROOF. By [3, Theorem 1, p. 1504], $Y=Y_0\cup\bigcup\{Y_i:i=1,2,\cdots\}$ with $f^{-1}(y)$ compact for $y\in Y_0$ and each Y_i discrete as a collection of singletons. Applying Lemma 1, we have a sequence $\{\mathscr{B}_i\}$ of locally finite open collections in Y with $\mathscr{B}=\bigcup\{\mathscr{B}_i:i=1,2,\cdots\}$ a base for the points of Y_0. Using collectionwise normality of Y, for each i swell the points y of Y_i to a discrete collection $\mathscr{E}_i=\{E_{i,y}:y\in Y_i\}$ of open sets with $y\in E_{i,y}$. By Lemma 5, each point y of Y_i has a closure preserving open base of neighborhoods $\mathscr{D}_y$. We may assume without loss of generality that the elements of $\mathscr{D}_y$ are subsets of $E_{i,y}$. Setting $\mathscr{F}_i=\bigcup\{\mathscr{D}_y:y\in Y_i\}$ we obtain a closure preserving base for the points of Y_i. Consequently

$$\bigcup\{\mathscr{B}_i:i=1,2,\cdots\}\cup\bigcup\{\mathscr{F}_i:i=1,2,\cdots\}$$

is a σ-closure preserving base for Y and Theorem 6 is proved.

Recall that the closed images of metric spaces have been characterized by Lašnev [4]. For such spaces, called Lašnev spaces, the implications metric space→Lašnev space→M_1 space hold. Moreover, as an immediate consequence of Theorem 6 we obtain in Corollary 7 a partial solution to Ceder's problem of whether every subspace of an M_1 space is an M_1 space.

COROLLARY 7. *Every subspace of a Lašnev space is* M_1.

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