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ZARISKI'S THEOREM ON SEVERAL LINEAR SYSTEMS

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ABSTRACT. We give a modern and fairly easy proof of (a slight improvement of) an important theorem of Zariski. The result gives conditions under which certain multigraded rings and modules associated with n linear systems are finitely generated, in a very strong sense.

Suppose L is a line bundle on a complete scheme X and R is a graded subring of $\bigoplus_{\nu\geqq 0} H^0(X, L^\nu)$ whose degree one part generates L. Then $\bigoplus_{\nu\geqq 0} H^0(X, L^\nu)$ is a finitely generated R module. Zariski has given a very useful souped-up version of this fact, working with several line bundles simultaneously [**1**, 5.1]. Since his proof is difficult for newly educated geometers to follow, it seems worthwhile to give a modern proof. That is the only purpose of this paper.

Before we state our slight improvement of Zariski's theorem, we must make some definitions. By an "m-fold graded ring," we mean a ring G together with a direct sum decomposition $G=\bigoplus\{G_\alpha:\alpha\in Z^m\}$ such that the multiplication map factors through maps $G_\alpha\otimes G_\beta\to G_{\alpha+\beta}$. We let $e_i\in Z^m$ be the element with 1 in the ith place and zeroes elsewhere. Let G' be the sub-G_0 algebra of G generated by terms of total degree 1.

1. DEFINITION. Let G be an m-fold graded ring, M a graded G module, and i an integer between 1 and m. Then M is "i-finite" if for some integer n, the maps $G_{e_i}\times M_\alpha\to M_{\alpha+e_i}$ are surjective whenever $\alpha_i\geqq n$. If M is i-finite for all i, we say M is "polyfinite."

2. PROPOSITION. *If M is finitely generated as a G' module, it is polyfinite. The converse holds if we assume that each M_α is finitely generated as a G_0 module and that $M_\alpha=0$ for any $\alpha_i\ll 0$.*

PROOF. First prove the following easy statements:

2.1 If M is i-finite, so is the shifted module $M(\alpha)$ for any $\alpha\in Z^m$.

2.2 If M and N are i-finite, so is $M\oplus N$.

2.3 A quotient of an i-finite module is i-finite.

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Now it is clear that G' is polyfinite as a module over itself, since all the maps $G'_{e_i} \times G'_\alpha \to G'_{\alpha+e_i}$ are surjective, if $\alpha_i \geqq 0$. Moreover, any finitely generated graded G' module M is a quotient of a finite direct sum of modules $G'(\beta)$, one for each generator of degree $-\beta$. Thus, it is polyfinite as a G' module and hence as a G module.

To prove the converse, we see that if M is polyfinite, there is a $\beta \in \mathbf{Z}^m$ such that the maps $G_{e_i} \times M_\alpha \to M_{\alpha+e_i}$ are surjective if $\alpha_i \geqq \beta_i$. Then we easily see that $\bigoplus_{\alpha \leqq \beta} M_\alpha$ generates M as a G' module. Since $\bigoplus_{\alpha \leqq \beta} M_\alpha$ is finitely generated as a G_0 module, it generates a finite G' module, and the proof is complete. $\square$

We can now state our version of Zariski's theorem:

3. THEOREM. *Let F be a coherent sheaf on a scheme X, proper over a field k, and let $L_1, \cdots, L_m$ be line bundles on X. Let Γ be an m-fold graded subring of $\bigoplus \{H^0(X, L_1^{\alpha_1} \otimes \cdots \otimes L_m^{\alpha_m}) : \alpha \in \mathbf{Z}^m$ and $\alpha \geqq 0\}$, and let M be a graded Γ submodule of $\bigoplus_{\alpha \geqq 0} H^n(X, F \otimes L_1^{\alpha_1} \otimes \cdots \otimes L_m^{\alpha_m})$. If the linear system Γ_{e_i} has no base points for each i, then M is polyfinite.*

Instead of proceeding directly with the proof of this theorem, we first consider what is essentially the universal case.

4. PROPOSITION. *Let k be a field, $V_1, \cdots, V_m$ finite dimensional vector spaces over k, and $Z = P(V_1) \times \cdots \times P(V_m)$. If F is a coherent sheaf on Z and if $\alpha \in \mathbf{Z}^m$, let $F(\alpha)$ be $F \otimes p_1^*(O_{P(V_1)}(\alpha_1)) \otimes \cdots \otimes p_m^*(O_{P(V_m)}(\alpha_m))$, where $p_i : Z \to P(V_i)$ is the projection. Then*:

4.1 *The natural map*: $G = S^\cdot(V_1) \otimes_k \cdots \otimes_k S^\cdot(V_m) \to \bigoplus_\alpha H^0(Z, O_Z(\alpha))$ *is an isomorphism of m-fold graded rings.*

4.2 $H^q(Z, O_Z(\alpha)) = 0$ *if $q > 0$ and $\alpha \geqq 0$.*

4.3 $\bigoplus \{H^q(Z, F(\alpha)) : \alpha \in \mathbf{Z}^m$ *and* $\alpha \geqq 0\}$ *is a finitely generated G module, for all q.*

4.4 *If $q > 0$, $H^q(Z, F(\alpha)) = 0$ for all $\alpha \gg 0$.*

PROOF. If $m = 1$, this is Serre's theorem [**2**, p. 47]. We shall prove 4.1, 4.2, and 4.3′ by induction on m, where 4.3′ is the statement 4.3 for F of the form $O_Z(\beta)$ for some $\beta \in \mathbf{Z}^m$. Assuming them proved for m and for Z with the same notation, we let V be another vector space and prove them for $Z \times P(V)$. In the diagram, all the maps are the natural ones. If

$$\begin{array}{ccc} Z \times P(V) & \xrightarrow{g} & P(V) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ Z & \xrightarrow{p} & \operatorname{Spec} k \end{array}$$

$\alpha \in \mathbf{Z}^m$ and $\nu \in \mathbf{Z}$, then $O_{Z \times P}(\alpha, \nu) = f^* O_Z(\alpha) \otimes g^* O_P(\nu)$. By the base change formula, the natural map: $O_Z(\alpha) \otimes R^q f_* g^* O_P(\nu) \to R^q f_* O_{Z \times P}(\alpha, \nu)$

is an isomorphism. Since our diagram is Cartesian and p is flat, the natural map: $p^*R^qh_*O_P(\nu)\to R^qf_*g^*O_P(\nu)$ is an isomorphism. Combining these with the base change formula for p, we get a natural isomorphism: $H^p(Z, O_Z(\alpha))\otimes_k H^q(P, O_P(\nu))\to H^p(Z, R^qf_*O_{Z\times P}(\alpha, \nu))$. By the induction hypothesis the map:

$$G_\alpha\otimes S^\nu(V)\to H^0(Z, O_Z(\alpha))\otimes_k H^0(P, O_P(\nu))\cong H^0(Z\times P, O_{Z\times P}(\alpha, \nu))$$

is an isomorphism, so 4.1 is proved. By induction, if $(\alpha, \nu)\geqq 0$, we see that $H^p(Z, R^qf_*O_{Z\times P}(\alpha, \nu))=0$ if p or $q>0$, so by the Leray spectral sequence, $H^i(Z\times P, O_{Z\times P}(\alpha, \nu))=0$ if $i>0$, and 4.2 is proved. Finally, for any β and μ, $\oplus_{\alpha\geqq 0} H^p(Z, O_Z(\beta+\alpha))$ is finite as a G module and

$$\bigoplus_{\nu\geqq 0} H^q(P, O_P(\mu+\nu))$$

is finite as an $S^\cdot(V)$ module, by the induction hypothesis; so their tensor product $\oplus_{(\alpha,\nu)\geqq 0} H^p(Z, R^qf_*O_{Z\times P}(\beta+\alpha, \mu+\nu))$ is finite as a $G\otimes_k S^\cdot(V)$ module. Consequently the abutment $\oplus_{(\alpha,\nu)\geqq 0} H^i(Z\times P, O_{Z\times P}(\beta+\alpha, \mu+\nu))$ is also finitely generated, so 4.3′ is also proved.

To finish the proof, we recall that the Segre embedding [3, p. 93] shows that the sheaf $L=O_Z(1, \cdots, 1)$ is very ample on Z. Therefore any coherent F on Z is a quotient of a finite direct sum E of copies of L^ν, for some ν. Moreover, 4.3 and 4.4 are proved for E, and also for all q sufficiently large, since $H^q(Z, \ )=0$ for $q\gg 0$. Now if $0\to K\to E\to F\to 0$ is exact then we get exact sequences $H^q(Z, E(\alpha))\to H^q(Z, F(\alpha))\to H^{q+1}(Z, K(\alpha))$. Then the theorem for E and a descending induction hypothesis on q will imply our result for F. $\square$

The proof of Theorem 3 is now quite easy. Let V_i be the (finite dimensional) k vector space Γ_{e_i}. Since V_i has no basepoints, there is a map $f_i: X\to P(V_i)$ such that $f_i^*O_{P(V_i)}(1)=L_i$. Then if $f: X\to Z$ is the induced map, $f^*O_Z(\alpha)=L_1^{\alpha_1}\otimes\cdots\otimes L_m^{\alpha_m}=L^\alpha$. Since f is proper, the sheaves R^qf_*F are coherent on Z. Hence the G module $\oplus_{\alpha\geqq 0} H^p(Z, R^qf_*F(\alpha))$ is finitely generated, and so is the abutment $\oplus_{\alpha\geqq 0} H^n(X, F(\alpha))$. Since G is noetherian, the G submodule M is also finitely generated. Finally, we note that $G_{e_i}=\Gamma_{e_i}$, so that by Proposition 2, M is polyfinite as a Γ module. This completes the proof. $\square$

5. Corollary. *Let H be ample on a projective scheme X, let L be a line bundle on X generated by its global sections, and let F be any coherent O_X module. Then there exists an integer J such that $H^q(X, F\otimes L^i\otimes H^j)=0$ if $q>0$, $i\geqq 0$, and $j\geqq J$.*

Proof. Suppose H^n is very ample, so that if $\Gamma=\oplus_{i,j} H^0(S, L^i\otimes H^{jn})$, Γ satisfies the hypothesis of Theorem 3. We apply the theorem with

$F \otimes H^m$ in place of F, where $0 \leqq m < n$, and conclude that each Γ module $\bigoplus_{i,j} H^q(X, F \otimes L^i \otimes H^{jn+m})$ is 1-finite. Hence there exists an integer I, independent of j, such that the map:

$$H^0(X, L) \otimes H^q(X, F \otimes L^{i-1} \otimes H^{jn+m}) \to H^q(X, F \otimes L^i \otimes H^{jn+m})$$

is surjective if $i \geqq I$, $j \geqq 0$, and $0 \leqq m < n$. Since H^n is ample we can find J such that $H^q(X, F \otimes L^i \otimes H^{jn+m}) = 0$ if $q > 0$, $j \geqq J$, $0 \leqq m < n$, and $0 \leqq i \leqq I$, and it follows immediately by induction on i that $H^q(X, F \otimes L^i \otimes H^j) = 0$ if $i \geqq 0$ and $j \geqq n(J+1)$.

REMARK. Zariski has proved [**1**, 6.2] that if $H^0(X, L)$ has only finitely many base points, then $H^0(X, L^i)$ has no base points for i sufficiently large, so we could weaken the hypothesis of the Corollary. His proof makes essential use of Theorem 3, but since it is quite readable, I have not included it here. I wish to thank the referee for filling a gap in my proof of Corollary 5.

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