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COUNTABLE TORSION PRODUCTS OF ABELIAN p-GROUPS

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ABSTRACT. Let $A_1, A_2, \cdots$ be direct sums of cyclic p-groups, and $G=t\prod_1^\infty A_n$ be the torsion subgroup of the product. The product decomposition is used to define a topology on G in which each $G[p^r]$ is complete and the restriction to $G[p^r]$ of any endomorphism is continuous. Theorems are then derived dual to those proved by Irwin, Richman and Walker for countable direct sums of torsion complete groups. It is shown that every subgroup of $G[p]$ supports a pure subgroup of G, and that if $G=H\oplus K$, then $H\approx t\prod_1^\infty G_n$, where the G_n are direct sums of cyclics. A weak isomorphic refinement theorem is proved for decompositions of G as a torsion product. Finally, in answer to a question of Irwin and O'Neill, an example is given of a direct decomposition of G that is not induced by a decomposition of $\prod_1^\infty A_n$.

1. **The product topology.** Let $G=t\prod_1^\infty A_n$ be the torsion subgroup of $\prod_1^\infty A_n$, where the A_n are abelian p-groups. For each nonnegative integer n, let $G_n=t\prod_{i>n}^\infty A_i$, considered as a subgroup of G in the natural way. The *product topology* on G is defined by taking the G_n to be a neighborhood basis at 0. Thus the product topology is the topology that $t\prod_1^\infty A_n$ inherits as a subspace of the cartesian product of the A_n, where these are considered discrete. Notice that, for each r, $G[p^r]$ is complete in the product topology.

We say that the A_n are *well-arranged* if the jth Ulm invariant of A_n is 0 whenever $j<n$. If the A_n are well-arranged, then for each r the product topology on $G[p^r]$ is finer than the p-adic topology. If the A_n are also bounded, then G is torsion complete and the product topology and p-adic topology coincide on $G[p^r]$.

Suppose now that each A_n is $\sum$-cyclic (a direct sum of cyclic groups). Let $A_n=\sum_{j=1}^\infty B_{nj}$, where B_{nj} is a direct sum of cyclic groups of order p^j. Then $G=t\prod_1^\infty A_n=t\prod_1^\infty A'_n$, with $A'_n=\sum_{j\geqq n} B_{nj}\oplus\prod_{j>n} B_{jn}$. Since A'_n is also $\sum$-cyclic, there is thus no loss of generality in supposing that the A_n are well-arranged. Notice that the decompositions in terms of the A_n and in terms of the A'_n have a common refinement.

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THEOREM 1. *Let $A_1, A_2, \cdots$ be well-arranged and let $B_1, B_2, \cdots$ be $\sum$-cyclic p-groups. Let ϕ be a homomorphism from $G=t\prod_1^\infty A_n$ to $t\prod_1^\infty B_n$. Then for each r, the restriction of ϕ to $G[p^r]$ is continuous in the product topology.*

PROOF. Consider first the case where $\phi: t\prod_1^\infty A_n \to B$ and B is $\sum$-cyclic. Then $B=\bigcup_1^\infty B_j$ where the B_j are summands of B and the nonzero elements of B_j have height at most j. We have $G[p^r]=\bigcup_j G[p^r]\cap\phi^{-1}(B_j)$. Since the $\phi^{-1}(B_j)$ are p-adically closed in G, they are also closed in the product topology (because the A_n are well-arranged). Since $G[p^r]$ is complete, from the Baire Category Theorem we see that, for some j, $G[p^r]\cap\phi^{-1}(B_j)$ is open in $G[p^r]$ in the product topology. This means that, for sufficiently large n, $\phi(G_n[p^r])\subseteq B_j$. Since ϕ is height increasing, this gives $\phi(G_n[p^r])=0$ if $n>j+r$. This means that the restriction of ϕ to $G[p^r]$ is continuous if B has the discrete topology.

The theorem now follows by applying this observation to the composition of $\phi: t\prod_1^\infty A_n \to t\prod_1^\infty B_n$ with the projection of $t\prod_1^\infty B_n$ onto B_n.

If we apply this theorem to a homomorphism $\phi: t\prod_1^\infty A_n \to B$, where each A_n is a direct sum of groups isomorphic to $Z(p^n)$ and B is $\sum$-cyclic, we recover the theorem that homomorphisms from torsion complete groups into $\sum$-cyclic groups are small [**6**].

2. **Socles.** At this point we need to recall some facts about the socles of p-groups without elements of infinite height. This material is fairly well known and is partially treated in [**1**].

By a *socle*, we mean a vector space V over Z/pZ, together with a function h which associates to each element in V a nonnegative integer or the symbol ∞, such that $h(x+y)\geqq\min\{h(x), h(y)\}$ and $h(0)=\infty$. It follows that $h(x+y)=h(x)$ whenever $h(x)<h(y)$. In particular, if G is a p-group, then $G[p]$ together with the usual p-height function h is a socle. A linear map $\phi: V\to W$ is a *morphism in the category of socles* if $h(\phi(x))\geqq h(x)$ for each $x\in V$.

If Y is a socle and X a subspace of Y, we give Y/X a socle structure by defining $h(y+X)=\sup\{h(y+x)|x\in X\}$. We say that a socle V is *homogeneous* of height n if $h(x)=n$ for all $0\neq x\in V$. We say that V is *summable* if $V=\sum_0^\infty V_n$ where each V_n is homogeneous of height n.

The following lemmas give some basic properties of socles. We have only sketched the proofs. For more details, see [**1**].

LEMMA 1. *Let V be a socle.*

(i) *If V is summable, then V is projective in the sense that whenever X is a subspace of a socle Y, then every morphism from V into Y/X factors through Y.*

(ii) *Let* $V=\sum V_n$ *where the* V_n *are subsocles of* V *and the sum is direct in the category of vector spaces. Then the sum is direct in the category of socles if and only if whenever* $x=\sum x_n$ *with* $x_n \in V_n$, *then* $h(x)=\min\{h(x_n)\}$.

(iii) *If* $h(x)$ *is bounded for* $0\neq x \in V$, *then* V *is summable.*

(iv) (*Kulikov's criterion for socles.*) V *is summable if and only if* $V=\bigcup_0^\infty W_n$ *where* $W_1 \subseteq W_2 \subseteq \cdots$ *is a chain of subspaces of* V *with* $h(x)\leqq n$ *whenever* $0\neq x \in W_n$.

(v) *If* V *is summable, and there is a monomorphism from* U *into* V, *then* U *is summable.*

PROOF. It is sufficient to prove (i) when V is homogeneous, and in this case it is clear. For (ii), suppose V is the vector space direct sum of V_1 and V_2. The projections of V onto V_1 and V_2 will be morphisms if $h(x)\geqq h(x+y)$ and $h(y)\geqq h(x+y)$ whenever $x \in V_1$ and $y \in V_2$. Since $h(x+y)\geqq \min\{h(x), h(y)\}$, this is equivalent to $h(x+y)=\min\{h(x), h(y)\}$. For (iii), suppose $h(x)\leqq n$ whenever $0\neq x \in V$, and let V_n be the subspace of V consisting of those $x \in V$ with $h(x)=n$, together with 0, and let U_n be a vector space complement. Then $V=U_n \oplus V_n$ and $h(x)<n$ for $x \in U_n$, so an induction gives the result. The necessity in (iv) is clear. Conversely, if $V=\bigcup_0^\infty W_n$ with each W_n bounded, then W_n/W_{n-1} is bounded and hence projective, so that W_{n-1} is a summand of W_n and V is a direct sum of summable socles and hence summable. From (iv), we immediately get (v).

LEMMA 2. *Let* G *be a* p*-group.*

(i) *If* $\{H_\alpha\}$ *if a family of pure subgroups of* G, *then the sum* $\sum H_\alpha$ *is direct and pure in* G *if and only if* $\sum H_\alpha[p]$ *is a direct sum in the category of socles.*

(ii) *If* $S\subseteq G[p]$ *and* S *is summable, then there is a pure* $\sum$*-cyclic subgroup* H *of* G *such that* $H[p]=S$.

(iii) G *is* $\sum$*-cyclic if and only if* $G[p]$ *is summable.*

PROOF. The necessity in (i) is clear. Conversely, if $\sum H_\alpha[p]$ is direct then so is $\sum H_\alpha$, since any nontrivial relation $\sum x_\alpha=0$, with $x_\alpha \in H_\alpha$, would lead to a nontrivial relation $\sum y_\alpha=0$, with $y_\alpha \in H_\alpha[p]$, after multiplication by a suitable power of p. The purity of $\sum H_\alpha$ in G follows easily from part (ii) of Lemma 1. Now if $S\subseteq G[p]$ and $S=\sum_0^\infty S_n$ where S_n is homogeneous of height n, then it is easily seen that there exist homogeneous pure subgroups H_n in G with $H_n[p]=S_n$. Then by (i), $H=\sum_0^\infty H_n$ is a pure $\sum$-cyclic subgroup of G, and $H[p]=S$. The necessity of (iii) is clear, and the sufficiency follows by applying (ii) to $S=G[p]$.

3. **Pure completeness.** We can now prove theorems dual to those proved by Irwin, Richman, and Walker [3]. We begin by showing that if G is a countable torsion product of $\sum$-cyclic p-groups, then G is pure complete. (For the dual theorem, see [5].)

THEOREM 2. *Let $G=t\prod_1^\infty A_n$ where the A_n are $\sum$-cyclic and well-arranged. Let $S\subseteq G[p]$. Then there is a pure subgroup H of G such that $H[p]=S$. Moreover, if S is closed in the product topology, then we can choose H so that $H\approx t\prod_1^\infty C_n$, where the C_n are pure $\sum$-cyclic subgroups of G.*

PROOF. Let $S_n=S\cap G_n$. We have a natural monomorphism from S_n/S_{n+1} into $(G/G_{n+1})[p]$. But $(G/G_{n+1})[p]$ is summable, since $G/G_{n+1}\approx\sum_1^n A_i$ is $\sum$-cyclic. By Lemma 1, S_n/S_{n+1} is summable and hence projective, so $S_n=R_n\oplus S_{n+1}$. Since $R_n\approx S_n/S_{n+1}$ is summable, by Lemma 2 there is a pure $\sum$-cyclic subgroup C_n of G_n with $C_n[p]=R_n$. Let $B=\sum_1^\infty C_n$. By Lemma 2, this sum is direct and B is pure in G.

Let K be the closure of B in the product topology. Then K consists of those elements of the form $\sum_1^\infty c_i$ with $c_i\in C_i$ and $o(c_i)$ bounded. Such sums necessarily converge, since $C_n\subseteq G_n$, so $K\approx t\prod_1^\infty C_n$, and $K[p]$ is the closure of S in the product topology. Let k be an element of $K[p]$. Then $k=\sum_1^\infty c_n$, where $c_n\in C_n[p]=R_n$. The height of k in K is the minimum of the heights of the c_n. We claim that this is also the height of k in G. Choose m so that $h(k)=h(\sum_1^m c_n)$ and $\min\{h(c_n)\mid n\leqq m\}=\min\{h(c_n)\}$. Then $h(k)=h(\sum_1^m c_n)=\min\{h(c_n)\}$, establishing the claim and so showing that K is pure.

If S is closed in the product topology, then $S=K[p]$ and, setting $H=K$, we are done. Otherwise, let H be a subgroup of K containing B and maximal with respect to the property $H[p]=S$. Since B is a basic subgroup of K, it easily follows that H is pure.

The main theorem now follows easily.

THEOREM 3. *If H is a summand of $G=t\prod_1^\infty A_n$, then $H\approx t\prod_1^\infty C_n$ where the C_n are $\sum$-cyclic pure subgroups of G.*

PROOF. $H[p]$ is p-adically closed and hence closed in the product topology if the A_n are well-arranged. By Theorem 2, there is a pure subgroup K of G, having the desired form, such that $H[p]=K[p]$. But since H is a summand, it follows that $H\approx K$ [**4**, Theorem 16].

4. **Isomorphic refinement.** In [**3**], it is shown that if a group G is a countable direct sum of torsion complete groups, then any two direct sum decompositions of G have isomorphic refinements. The dual theorem for torsion products of $\sum$-cyclic groups fails, as is shown by the following example.

EXAMPLE. Let $G=\prod_1^\infty A_n=B_1\oplus B_2$, where $A_n\approx Z(p)$ for each n, and $B_1=\sum_1^\infty A_n$. Then it is clear that we cannot have $G=\prod_1^\infty E_n\oplus\prod_1^\infty F_n$ with $E_n\oplus F_n\approx A_n$ and $\prod_1^\infty E_n\approx B_1$.

We do have a weaker theorem, however, involving a chain of refinements. The main results are stated in the following two lemmas.

LEMMA 3. *Let $G=t\prod_1^\infty A_n=t\prod_J B_j$ where the A_n and B_j are $\sum$-cyclic. Then there is a countable partition $\{J(n)|n=1,2,\cdots\}$ of J such that $C_n=\prod_{J(n)} B_j$ is $\sum$-cyclic and $G=t\prod_1^\infty C_n$.*

PROOF. We may suppose that the A_n are well-arranged. By applying Theorem 1 to the projection ψ_j of G onto B_j, we see that for each j there is an n such that $\psi_j(G_n[p])=0$ where $G_n=t\prod_{i>n} A_i$. For fixed n, let $J(n)$ be the set of $j\in J$ such that $\psi_j(G_n[p])=0$ and $\psi_j(G_{n-1}[p])\neq 0$. Let $C_n=t\prod_{J(n)} B_j$. We have $G_n[p]\subseteq\bigcap\{\operatorname{Ker}\psi_j|j\in J(m),\ m\leqq n\}=t\prod_{i>n} C_i$. Hence $C_n\cap G_n=0$, so C_n can be embedded in $\prod_1^n A_i$ and is thus $\sum$-cyclic. This is only possible if B_j is bounded for all but finitely many $j\in J(n)$, so $C_n=t\prod_{J(n)} B_j=\prod_{J(n)} B_j$.

LEMMA 4. *Let $G=t\prod_1^\infty A_n=t\prod_1^\infty B_n$ where the A_n and B_n are $\sum$-cyclic and well-arranged. Then there is a family of $\sum$-cyclic groups F_{ij} having the following properties:*

(i) *For each n, only finitely many F_{in} and F_{nj} are nonzero.*

(ii) *For each n, $A_n\approx\sum_j F_{nj}$ and $B_n\approx\sum_i F_{in}$.*

PROOF. By Theorem 1, applied to the projection of G onto B_1, $G_m[p]=t\prod_{i>m} A_i[p]\subseteq t\prod_{j\geqq 2} B_j$ for some m. By Theorem 2, applied to $t\prod_{j\geqq n} B_j$, there is a pure subgroup $H\subseteq t\prod_{j\geqq 2} B_j$ with $G_m[p]=H[p]$, and by [4, Theorem 16] we have $G=\prod_1^m A_i\oplus H$. Hence we may suppose $G_m\subseteq t\prod_{j\geqq 2} B_j$. It is then easily seen that $\prod_1^m A_i=\phi(B_1)\oplus\phi(t\prod_{j\geqq 2} B_j)$, where ϕ is the projection onto $\prod_1^m A_i$. For convenience, we will replace B_1 by $\phi(B_1)$. Since $\prod_1^m A_i$ is $\sum$-cyclic, we can write $B_1=\sum_1^m F_{i1}$, $\phi(t\prod_{j\geqq 2} B_j)=\sum_1^m A_i'$ where $F_{i1}\oplus A_i'\approx A_i$. Set $F_{i1}=0$ for $i>m$. We have $t\prod_{j\geqq 2} B_j=\sum_1^m A_i'\oplus t\prod_{i>m} A_i$, and we now apply the same procedure to A_1'. We continue in this way, criss-crossing from the A's to the B's, constructing the desired groups F_{ij}.

Summarizing, we get the following theorem.

THEOREM 4. *Let $G=t\prod_1^\infty A_n=t\prod_J B_j$, with the A_n $\sum$-cyclic.*

(i) *We can write $G=t\prod_1^\infty C_n$ where the C_n are $\sum$-cyclic and the products in terms of the B_j and C_n have a common refinement.*

(ii) *We can write $G=t\prod_1^\infty A_n'=t\prod_1^\infty C_n'$ where the A_n' and C_n' are $\sum$-cyclic and well-arranged. The products in terms of the A_n and A_n' have a common refinement, and likewise for the C_n and C_n'. The products in terms of the A_n' and C_n' have an isomorphic refinement.*

PROOF. By Theorem 3, $t\prod_J B_j$ can be refined to a torsion product of $\sum$-cyclic groups. By Lemma 3, this product is a refinement of a countable torsion product of $\sum$-cyclic groups. Now if $A_n=\sum_j A_{nj}$ with A_{nj} a direct sum of cyclic groups of order p^j, define $A'_n=\sum_{j\geq n} A_{nj}\oplus\prod_{j>n} A_{jn}$. The C'_n are similarly defined. By Lemma 4, the products in terms of the A'_n and the C'_n have an isomorphic refinement.

5. **Additional results.** We present here two additional theorems concerning countable torsion products of arbitrary abelian p-groups.

Let A_n be p-groups and let $B_n=\sum_{j=1}^{\infty} B_{nj}$ be a basic subgroup of A_n, with B_{nj} a direct sum of cyclic groups of order p^j. Irwin and O'Neill **[2]** have pointed out that $\sum_{j=1}^{\infty}\prod_{n=1}^{\infty} B_{nj}$ is a basic subgroup of $t\prod_1^{\infty} A_n$.

THEOREM 5. *The p-adic closure of $\sum_1^{\infty} A_n$ in $t\prod_1^{\infty} A_n$ is a summand.*

PROOF. With B_n and B_{nj} as above, for each j choose $K_j\subseteq\prod_{n>j} B_{nj}$ such that $\prod_{n=1}^{\infty} B_{nj}=\sum_{n=1}^{\infty} B_{nj}\oplus K_j$. Let C be the p-adic closure of $\sum_1^{\infty} A_n$ in $t\prod_1^{\infty} A_n$ and let K be the closure in the product topology of $\sum_1^{\infty} K_j$. The elements of K have the form $\sum_1^{\infty} x_j$ with $x_j\in K_j$ and $o(x_j)$ bounded. Such sums always converge in the product topology since $K_j\subseteq\prod_j^{\infty} A_n$. Hence $K\approx t\prod_1^{\infty} K_j$. Since the K_j are bounded, K is torsion complete. Hence the projection of $\sum_j\prod_n B_{nj}$ onto $\sum K_j$ extends uniquely to a projection of $t\prod_1^{\infty} A_n$ onto K. The kernel of this projection is the p-adic closure of $\sum B_n$, which is C. Hence $t\prod_1^{\infty} A_n=C\oplus K$.

The next theorem applies to $\prod_1^{\infty} A_n$. By suitably topologizing $\prod_1^{\infty} A_n$, we get an analogue to Theorem 1, with no restriction on the A_n except that they be abelian p-groups.

THEOREM 6. *Let A_n be p-groups and let $\prod_1^{\infty} A_n$ be given the topology which is the supremum of the product topology and the p-adic topology. Then any endomorphism ϕ of $\prod_1^{\infty} A_n$ is continuous.*

PROOF. Let ϕ_k be the composition of ϕ with the projection onto A_k. It suffices to show that each ϕ_k is continuous, where A_k is discrete. Now $\prod_1^{\infty} A_n$ is a complete metrizable space, and we have $\prod_1^{\infty} A_n=\bigcup_r \phi_k^{-1}(A_k[p^r])$. The Baire Category Theorem then yields $\phi_k(p^m\prod_{n\geq m} A_n)\subseteq A_k[p^r]$ for some r and sufficiently large m, and consequently

$$\phi_k\left(p^{m+r}\prod_{n\geq m} A_n\right)=0.$$

Irwin and O'Neill **[2]** have asked whether every direct decomposition of $t\prod_1^{\infty} A_n$ is induced by a decomposition of $\prod_1^{\infty} A_n$. We now show that this is not the case by constructing a projection on $t\prod_1^{\infty} A_n$ that cannot be extended to $\prod_1^{\infty} A_n$.

Let $A_n=\sum_{j\geq n}\langle a_{nj}\rangle$, with $\langle a_{nj}\rangle\approx Z(p^j)$. Define ϕ on $\sum_1^{\infty} A_n$ by $\phi(a_{2n,2n})=a_{2n,2n}+a_{1,n}$, and $\phi(a_{ij})=0$ otherwise. Under the topology of Theorem 6,

ϕ is continuous on $\sum_n A_n[p^r]$ for each r and $\prod_n A_n[p^r]$ is complete, so ϕ extends uniquely to $t\prod_1^\infty A_n$. Since ϕ is a projection, so is its extension to $t\prod_1^\infty A_n$. But ϕ is not continuous on $t\prod_1^\infty A_n$, because $p^{n-1}a_{2n,2n}$ converges to 0 whereas $\phi(p^{n-1}a_{2n,2n})=p^{n-1}a_{2n,2n}+p^{n-1}a_{1,n}$ does not. Thus ϕ cannot be extended to $\prod_1^\infty A_n$.

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