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APPLICATION OF LIAPUNOV THEORY TO BOUNDARY VALUE PROBLEMS. II[1]

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ABSTRACT. The second order vector differential equation $x''=f(t,x,x')$ is considered. Liapunov conditions for existence of solutions to boundary value problems are obtained strictly in terms of the function f, extending the results of George and Sutton [1].

1. **Introduction.** In [1], George and Sutton formulated a Liapunov theory for the existence and uniqueness of solutions to the following class of boundary value problems:

$$x'' = f(t, x, x'), \tag{1.1a}$$

$$x(a) = A, \quad x(b) = B, \qquad \|A\|, \|B\| \leqq M, M > 0, \tag{1.1b}$$

where A, B are fixed n-vectors, $'=d/dt$, x is a n-vector with a continuous second derivative on $[a, b]$, and the n-vector function f is defined and continuous on $D=[a, b]\times R^{2n}$ where $[a, b]$ is an interval on the real line. The development in [1] replaced certain conditions of Hartman [2] by Liapunov-type results. Fundamental to this approach was the construction of a function g which is equal to f on a bounded region. To describe in more detail Hartman's results, we shall need the concept of a Nagumo function, a continuous scalar function $\phi(s)$ with the properties

$$\phi(s) > 0 \quad \text{for } 0 \leqq s < \infty, \qquad \int_0^\infty \frac{s\,ds}{\phi(s)} = \infty. \tag{1.2}$$

Finally, $\langle x, f\rangle$ denotes the scalar product, and $\|x\|$ the norm in Euclidean space.

THEOREM 1.1 (HARTMAN [2]). *Let the function $f(t, x, x')$ be continuous on the set*

$$D_M = [a, b] \times \{x : \|x\| \leqq M\} \times R^n.$$

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Let f have the following properties:

$$\langle x, f\rangle + \|x'\|^2 \geqq 0 \quad \text{when } \langle x, x'\rangle = 0 \text{ and } \|x\| = M, \tag{1.3}$$

$$\|f\| \leqq \phi(\|x'\|), \qquad \|f\| \leqq 2\alpha(\langle x, f\rangle + \|x'\|^2) + K. \tag{1.4}$$

Here $R>0$, $K>0$, $\alpha\geqq 0$ are suitable constants and $\phi(s)$ is a Nagumo function (cf. (1.2)). *Let $\|A\|$, $\|B\|\leqq M$. Then* (1.1) *has at least one solution.*

PROOF. We will sketch the proof to motivate our results. To begin, we choose an N (depending only on ϕ, α, R, K) such that, for any function f satisfying (1.4) on D_M and any solution $x(t)$ of the corresponding differential equation (1.1a), the inequality $\|x(t)\|\leqq M$ for all $t\in[a, b]$ implies

$$\|x'(t)\| \leqq N \quad \text{(cf. [2, p. 433])}. \tag{1.5}$$

LEMMA 1.1. *There exists an n-vector function g defined on the set $D=[a, b]\times R^{2n}$ having the following properties*:

(i) *g is bounded and continuous on D.*

(ii) *$g(t, x, x')=f(t, x, x')$ when $a\leqq t\leqq b$, $\|x\|\leqq M$, $\|x'\|\leqq N$.*

(iii) *$\|g\|\leqq\phi(\|x'\|)$, $\|g\|\leqq 2\alpha(\langle x, g\rangle+\|x'\|^2)+K$ when $(t, x, x')\in D_M$.*

(iv) *$\langle x, g\rangle+\|x'\|^2>0$ when $\langle x, x'\rangle=0$ and $\|x\|>M$.*

The proof then proceeds by replacing condition (1.3) by:

$$\langle x, f\rangle + \|x'\|^2 > 0 \quad \text{when } \langle x, x'\rangle = 0 \text{ and } \|x\| = M. \tag{1.6}$$

Later, a continuity argument is used to get back to (1.3). Using condition (1.4), an N is selected as in (1.5), then a g is constructed as in Lemma 1.1. The differential equation

$$x'' = g(t, x, x') \tag{1.7}$$

is considered having the properties listed in Lemma 1.1. Solutions $x(t)$ of (1.7) have the property that if $\|x(t)\|\leqq M$, then $\|x'(t)\|\leqq N$ on $[a, b]$. Also (1.7) and (1.1b) have a solution $x(t)$ (cf. [**2**, p. 433]). Let $r(t)=\|x(t)\|^2$. Condition (iv) in Lemma 1.1 means that $r''>0$ if $r'=0$ and $r>M^2$. Now r can assume its maximum only where $r'=0$, $r''\leqq 0$ so $(r(t))^{1/2}=\|x(t)\|\leqq M$ on $[a, b]$.

By condition (ii) in Lemma 1.1, we are led to the following result: A solution of (1.7) and (1.1b) is a solution of (1.1).

A continuity argument using (1.6) to get (1.3) (cf. [**2**, p. 433]) and Arzela's theorem completes the proof of Theorem 1.

In [**1**], it was shown that conditions (iii) and (iv) in Lemma 1.1 could be replaced by Liapunov conditions resulting in the same conclusion to Theorem 1.1.

In this paper, a more restrictive class of Liapunov functions than those considered in [**1**] are introduced. This allows the existence theorems in [**1**]

to be proved with conditions in terms of f rather than g as in [**1**]. Thus, a principal difficulty in using the results of [**1**] is removed. Also, the new class of Liapunov functions is still sufficiently large to include Hartman's condition [**2**].

2. **Liapunov conditions for existence of solutions.** Let a Liapunov function $V(t, x, x')$ be continuously differentiable in (t, x, x'), a real valued function. Let

$$\frac{\partial V}{\partial x} = \left(\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_n}\right), \qquad \frac{\partial V}{\partial x'} = \left(\frac{\partial V}{\partial x_1'}, \cdots, \frac{\partial V}{\partial x_n'}\right).$$

Define the derivative of V with respect to t along a solution of (1.1a) to be

$$V_f' = \partial V/\partial t + \langle \partial V/\partial x, x'\rangle + \langle \partial V/\partial x', f\rangle.$$

Also let $x(t)$ belong to the class $C^2[a, b]$ if $x''(t)$ is continuous on $[a, b]$. We will now restate a lemma of George and Sutton [**1**] which is an alternative to (1.4) in terms of this class of Liapunov functions.

LEMMA 2.1 (GEORGE AND SUTTON [**1**]). *Suppose there exists a Liapunov function $U(t, x, x')$ defined on $D_M = [a, b] \times \{x : \|x\| \leqq M\} \times R^n$. Let $X(t)$ be the set of functions $x(t) \in C^2[a, b]$ satisfying $\|x(t)\| \leqq M$ on $[a, b]$, and also the following properties*:

(i) $U(a, x(a), x'(a)) = 0$.

(ii) $U(t, x, x') \geqq (t-a)\phi(\|x'\|)$, *where ϕ is a positive continuous function defined on $[0, \infty)$ such that $\phi(r) \to \infty$ as $r \to \infty$.*

(iii) *There exists a positive constant L such that*

$$U' = \partial U/\partial t + \langle \partial U/\partial x, x'\rangle + \langle \partial U/\partial x', x''\rangle \leqq L$$

in the interior of D_M.

Then there exists a positive constant N such that $\|x'(t)\| \leqq N$ on $[a, b]$, for every $x(t)$ belonging to $X(t)$.

EXAMPLE. Let $U = (t-a)\|x'\|^2$. Then conditions (i) and (ii) are satisfied. Condition (iii) becomes

$$\langle x'(t), x''(t)\rangle \leqq (L - \|x'(t)\|^2)/2(t-a) \quad \text{on } (a, b]. \tag{2.1}$$

From the lemma, there exists an N such that if (2.1) is satisfied and $\|x(t)\| \leqq M$ then $\|x'(t)\| \leqq N$ on $[a, b]$. Many lemmas similar to Lemma 2.1 could be developed. This was simply an attempt in [**1**] to show that alternatives existed to the standard Nagumo type theorems [**2**, pp. 428–429].

THEOREM 2.1. *Assume there exists a Liapunov function satisfying the hypothesis of Lemma 2.1. Let $V(t, x, x')$ be another Liapunov function*

defined on the set $D_1=[a, b]\times\{x:\|x\|\geqq M\}\times R^n$ *and satisfying*:

(i) $$\langle\partial V/\partial x', f\rangle \leqq 0 \quad \textit{for } \|x\| \geqq M,\ \|x'\| \geqq N+1,$$

and

(ii) $$\langle\partial V/\partial x', (M/\|x\|)f(t, Mx/\|x\|, x')\rangle \geqq \langle\partial V/\partial x', f\rangle \quad \textit{for } \|x\| \geqq M \textit{ and } \|x'\| \leqq N.$$

Then $V_f'\geqq 0$ *in the interior of* D_1 *implies* $V_g'\geqq 0$ *in the interior of* D_1.

PROOF. Let N be the constant provided by Lemma 2.1. As in Hartman [2]; let

$$\begin{aligned} g(t, x, x') &= 1 - \delta(\|x'\| - N)f(t, x, x') \quad \text{on } D_M, \\ &= (M/\|x\|)g(t, Mx/\|x\|, x') \qquad \text{when } \|x\| > M. \end{aligned}$$

Here,

$$\begin{aligned} \delta(s) &= 0 \quad \text{if } s < 0, \\ &= s \quad \text{if } 0 \leqq s \leqq 1, \\ &= 1 \quad \text{if } s > 1. \end{aligned}$$

The proof now consists of establishing that $V_g'\geqq 0$ on D_1 with $\|x\|=M$. Then, the argument is extended to $\|x\|>M$. The proof is motivated by a result of Knobloch [3].

On D_M with $\|x\|=M$,

$$V_g' = \partial V/\partial t + \langle\partial V/\partial x, x'\rangle + \langle\partial V/\partial x', f\rangle\tau \tag{2.2}$$

where $\tau=1-\delta(\|x'\|-N)\in[0, 1]$. Since (2.2) is a linear function of $\tau\in[0, 1]$, if $V_g'\geqq 0$ for $\tau=0$ and $\tau=1$, then $V_g'\geqq 0$ for all $\tau\in[0, 1]$. At $\tau=1$, $V_g'=V_f'\geqq 0$. At $\tau=0$, $\|x'\|>N+1$, we have, from condition (i),

$$\begin{aligned} V_g' &= \partial V/\partial t + \langle\partial V/\partial x, x'\rangle \geqq \partial V/\partial t + \langle\partial V/\partial x, x'\rangle + \langle\partial V/\partial x', f\rangle \\ &= V_f' \geqq 0. \end{aligned}$$

If $\|x\|>M$,

$$\begin{aligned} V_g' &= \frac{\partial V}{\partial t} + \left\langle\frac{\partial V}{\partial x}, x'\right\rangle + \left\langle\frac{\partial V}{\partial x'}, \frac{M}{\|x\|}\, g\left(t, \frac{Mx}{\|x\|}, x'\right)\right\rangle \\ &= \frac{\partial V}{\partial t} + \left\langle\frac{\partial V}{\partial x}, x'\right\rangle + \frac{M\tau}{\|x\|}\left\langle\frac{\partial V}{\partial x'}, f\left(t, \frac{Mx}{\|x\|}, x'\right)\right\rangle \geqq 0 \end{aligned}$$

by repeating the argument above for $\tau=0$ and $\tau=1$, using condition (ii) when $\tau=1$ and condition (i) when $\tau=0$.

Using this result, Theorem 4.2 in [1], which is a generalization of (1.3), can now be stated as follows:

THEOREM 2.2. *Let $x(t)$ be a solution of* (1.1a) *defined on* $[a, b]$ *and suppose that* $\|x(a)\|$, $\|x(b)\| \leqq M$. *Let there exist a Liapunov function* $V(t, x, x')$ *defined on* D_1 *such that*

(i) $V(t, x, x')=0$ *whenever* $\|x\|=M$,

(ii) $V(t, x, x')>0$ *whenever* $\|x\|>M$,

(iii) $\langle \partial V/\partial x', f\rangle \leqq 0$ *for* $\|x\| \geqq M$, $\|x'\| \geqq N+1$,

(iv) $\langle \partial V/\partial x', (M/\|x\|)f(t, Mx/\|x\|, x')\rangle \geqq \langle \partial V/\partial x', f\rangle$ *for* $\|x\|>M$, $\|x'\| \leqq N$,

(v) $V'_f \geqq 0$ *in the interior of* D_1.

Then $\|x(t)\| \leqq M$ *for all* $t \in [a, b]$.

PROOF. From Theorem 2.1, condition (v) can be replaced by $V'_g \geqq 0$ in the interior of D_1. The proof continues exactly the same as the proof of Theorem 4.2 in [**1**].

REMARK. The derivative V'_f in condition (v) will be evaluated along a solution of (1.1a) even if f does not explicitly appear in V'_f.

EXAMPLE 1. $V=\|x'\|^2(\|x\|^2-M^2)$ shows that $\partial V/\partial x$ need not be zero. In this case, (iii), (iv) and (v) will impose conditions on f.

EXAMPLE 2. By choosing $V(t, x, x')=\langle x, x\rangle - M^2$, $V'_f=2\langle x, x'\rangle$, $V''_f=2[\langle x, f\rangle + \|x'\|^2]$. Hartman's condition (1.5) implies V evaluated along a solution $x(t)$ of (1.1a) does not have a maximum at any point $t \in [t_1, t_2]$ when $\|x(t)\|>M$. This follows since if $V'_f=0$, $V''_f>0$. Now $V=0$ when $\|x\|=M$ and $V>0$ if $\|x\|>M$. If there existed a solution $x(t)$ such that $\|x(t)\|>M$, then $V'_f \geqq 0$ and all conditions of Theorem 2.2 are satisfied with the observation that $\partial V/\partial x'=0$. Thus, the more general Liapunov conditions include Hartman's condition (1.5) as a special case by a suitable V selection.

THEOREM 2.3. *Suppose the Liapunov function in Lemma* 2.1 *has the additional properties* $U'_f \leqq L$ *in the interior of* D_M *and*

(i) $$\langle \partial U/\partial x', f\rangle \geqq 0 \quad \textit{for } \|x\| \leqq M,\ \|x'\| \geqq N+1.$$

Then $U'_g \leqq L$ *in the interior of* D_M.

PROOF. In a similar fashion to Theorem 2.1,

$$U'_g = \partial u/\partial t + \langle \partial U/\partial x, x'\rangle + \langle \partial U/\partial x', f\rangle \tau.$$

At $\tau=1$, $U'_g=U'_f \leqq 0$. When $\tau=0$,

$$U'_g = \frac{\partial U}{\partial t} + \left\langle \frac{\partial U}{\partial x}, x' \right\rangle \leqq \frac{\partial U}{\partial t} + \left\langle \frac{\partial U}{\partial x}, x' \right\rangle + \left\langle \frac{\partial U}{\partial x'}, f \right\rangle = U'_f \leqq L,$$

by condition (i).

The results of [**1**] can now be stated entirely in terms of f.

THEOREM 2.4. *Suppose $f(t, x, x')$ is defined and continuous on $[a, b] \times R^{2n}$. Suppose there exists two Liapunov functions $V(t, x, x')$ and $U(t, x, x')$ as described in Theorems* 2.2 *and* 2.3. *Then the boundary value problem* (1.1) *has at least one solution.*

PROOF. From Theorems 2.2 and 2.3, we have $V'_g \geqq 0$ in the interior of D_1 and $U'_g \leqq L$ in the interior of D_M. The proof is now the same as for Theorem 4.4 in [**1**].

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