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SHORTER NOTES

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A NEW PROOF OF A REGULARITY THEOREM FOR ELLIPTIC SYSTEMS

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ABSTRACT. We give a proof, which makes use of the Riesz-Thorin theorem, for a smoothness theorem for solutions of elliptic systems in divergence form with bounded measurable coefficients. The results imply an important theorem in two dimensions due to Morrey [3]. Meyers has used a similar technique to get these results for elliptic equations [4].

Let Ω be a compact domain in R^n with smooth boundary and consider the $m\times m$ real elliptic system given by:

$$(Lu)_k=\sum_{i,j,l}\frac{\partial}{\partial x_i}a_{ij}^{lk}\frac{\partial}{\partial x_j}u_l$$

where a_{ij}^{lk} are bounded measurable functions on Ω satisfying the inequality

$$\sum_{i,j,k,l}\sum a_{ij}^{lk}\pi_l^i\pi_k^j\geqq\sum_{i,k}(\pi_k^i)^2.$$

$E^p=H_0^{1,p}(\Omega)$ is the usual complex Sobolev space of m complex functions with p-integrable derivatives which are 0 on the boundary of Ω, and $\| \ \|_p$ denotes norms in E^p or norms of operators on E^p.

THEOREM. *There exist $q_0>2$ depending on Ω, n and the number $K=\max|a_{ij}^{lk}-\delta_k^l\delta_j^i|$ such that $L\colon H_0^{1,q}(\Omega)\to H^{-1,q}(\Omega)$ is invertible for $q_0/(q_0-1)<q<q_0$.*

PROOF. Let Δ be the Laplace operator on Ω, Δ^{-1} its inverse.

$$\Delta^{-1}\colon H^{1,-p}(\Omega)\to H_0^{1,p}(\Omega)$$

has norm $c(p)<\infty$ for $1<p<\infty$ [1]. We define an analytic family of operators

$$A(\lambda)=\Delta^{-1}(L+\lambda\Delta)=\Delta^{-1}L+\lambda I.$$

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$A(\lambda)$ is a bounded linear operator on E^p for $1<p<\infty$. To prove the theorem it is sufficient to show that $A(0)$ is invertible on E^q for q in a neighborhood of 2.

$A(\lambda)$ is invertible in E^2 for Re $\lambda>-1$ since we have

$$\begin{aligned}\operatorname{Re}\langle u, A(\lambda)u\rangle_{E^2} &= \operatorname{Re}\lambda\langle u, \Delta u\rangle_{L^2} + \langle u, Lu\rangle_{L^2}\\ &\geqq (1+\operatorname{Re}\lambda)\langle \operatorname{grad} u, \operatorname{grad} u\rangle_{L^2}\end{aligned}$$

which shows that

$$\|A(\lambda)^{-1}\|_2 \leqq 1/(1+\operatorname{Re}\lambda).$$

We also have that $A(\lambda)$ is invertible on E^p for $|\lambda+1|>c(p)K$, since $\|[\lambda+1-A(\lambda)]u\|_p \leqq c(p)K\|u\|_p$, which shows that

$$\|A(\lambda)^{-1}\|_p \leqq 1/(|\lambda+1| - c(p)K).$$

We may apply an extension of the Riesz-Thorin theorem [2], with $\lambda=\lambda_1\alpha+\lambda_2(1-\alpha)$ (all real), $1/q=\alpha/2+(1-\alpha)/p$, $\lambda_2+1>c(p)K$ and $\lambda_1>-1$ to get that $A(\lambda)$ is invertible on E^q and

$$\|A(\lambda)^{-1}\|_q \leqq \left(\frac{1}{\lambda_2+1-c(p)K}\right)^{1-\alpha}\left(\frac{1}{1+\lambda_1}\right)^{\alpha}.$$

If we choose λ_1, λ_2, α and p properly, $\lambda=0$ and q lies in an interval about 2.

COROLLARY 1. *If $n=2$ and $Lu=f$ for $u \in H_0^{1,2}(\Omega)$ and $f\in L^2(\Omega)$, then u is Hölder continuous in Ω.*

PROOF. Use the Sobolev imbedding theorems. $f\in H^{-1,q}(\Omega)$ and $u\in H_0^{1,q}(\Omega)\subset C^\alpha(\Omega)$ for $(1-\alpha)/2=1/q<1/2$.

COROLLARY 2. *If K is sufficiently small, Corollary 1 holds if $n\neq 2$ and $f\in C^0(\Omega)$.*

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