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THE COMPACTNESS OF THE SET OF ARC CLUSTER SETS OF AN ARBITRARY FUNCTION

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ABSTRACT. It is known that if f is a continuous complex-valued function defined in the open unit disk D, then the set $\mathfrak{C}_f(\zeta)$ $(\zeta \in \partial D)$ of all arc cluster sets of f at ζ is compact in a natural topology for all but at most a countable number of points $\zeta \in \partial D$. We show that if f is an arbitrary complex-valued function defined on an arbitrary subset Z of the plane, then $\mathfrak{C}_f(p)$ is compact for all but at most a countable number of points $p \in Z \cup \partial Z$.

1. **Introduction.** Let $\mathfrak{H}$ be the collection of all closed, nonempty subsets of the Riemann sphere W. We define what is known as the Hausdorff metric on $\mathfrak{H}$ by

$$M(A, B) = \max\left(\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\right) \qquad (A, B \in \mathfrak{H}),$$

where $d(a, b)$ is the spherical distance between a and b. With this metric $\mathfrak{H}$ is a compact metric space.

Now let P denote the complex plane, and let $p \in P$. We say that α is an arc at p if $\alpha \subseteq P - \{p\}$ and is the image of a continuous function $z = z(t)$ $(0 \leqq t < 1)$ such that $z(t) \to p$ as $t \to 1$. We call α a simple arc at p if α is in addition homeomorphic to $[0, 1)$. If Z is any nonempty subset of P, we let $\bar{Z}$ denote the closure of Z and define, for $p \in \bar{Z}$,

$$\mathscr{A}_p(Z) = \{\alpha : \alpha \text{ is a simple arc at } p \text{ with } \alpha \subseteq Z\}.$$

We assume that the reader is familiar with the elementary notions of cluster set theory (see [3] or [4]). If f is an arbitrary function whose domain is Z and whose range is a subset of W, if $p \in \bar{Z}$ and $\alpha \in \mathscr{A}_p(Z)$, then $C(f, p, \alpha)$ denotes the arc cluster set of f at p along α. We let $\mathfrak{C}_f(p) = \{C(f, p, \alpha) : \alpha \in \mathscr{A}_p(Z)\}$. Then $\mathfrak{H}$ topologizes the set $\mathfrak{C}_f(p)$ with what has been called the M-topology.

This paper is written in response to the following two theorems which appear in [2, Theorems 1 and 2, pp. 211, 213].

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THEOREM A. *Let f be a continuous function in D, and let $\zeta \in K$. If ζ is not an ambiguous point of f, then $\mathfrak{C}_f(\zeta)$ is a compact set in the M-topology.*

THEOREM B. *There is a function f defined in D such that $\zeta_0=1$ is not an ambiguous point of f, and $\mathfrak{C}_f(1)$ is not a compact set in the M-topology.*

Since an arbitrary function in D can have at most a countable number of ambiguous points $\zeta \in K$ [**1**, Theorem 2, p. 380], it follows from Theorem A that if f is a continuous function in D, then $\mathfrak{C}_f(\zeta)$ is a compact set in the M-topology for all but at most a countable number of points $\zeta \in K$. This raises the question as to whether or not $\mathfrak{C}_f(\zeta)$ is a compact set in the M-topology for all but at most a countable number of points $\zeta \in K$ when f is an arbitrary complex valued function defined in D. But since the boundary of a domain plays no preferential role insofar as an arbitrary function is concerned, it is natural to investigate the question of compactness of $\mathfrak{C}_f(p)$ for points p interior to D as well. Although the result in this paper deals with an arbitrary domain, our most striking conclusion is that if f is an arbitrary function defined in the plane P, then $\mathfrak{C}_f(p)$ is a compact set in the M-topology for all but at most a countable number of points $p \in P$. Our result, in addition, answers the above question concerning the unit circle K in the affirmative.

2. **Geometry of arcs.** Our proof requires several rather specialized definitions. If α is an arc at $p \in P$, we say that β is a terminal subarc of α if $\beta=z([t_0, 1))$ for some t_0 and some representative $z=z(t)$ of α. We call γ *a curvilinear segment* if γ is the image of a continuous function $z=z(t)$ $(0\leqq t\leqq 1)$. *A sequence $\langle\gamma_j\rangle$ of curvilinear segments is said to converge to* $p \in P$ provided that $p \notin \gamma_j$ for every j, and that for every $\varepsilon>0$ all but a finite number of the sets γ_j are contained in the open disk of radius ε centered at p.

Let $E\subseteq P$. We say that Γ *is a selector of arcs on E* if for every $p \in E$, $\Gamma(p)$ is a nonempty collection of simple arcs at p, and if $\Gamma(p)=\varnothing$ for every $p \notin E$. If $Q\subseteq P$ and α, β are arcs at p, we say that *Q joins α and β* if there is a sequence $\langle\gamma_j\rangle$ of curvilinear segments such that $\gamma_j \subseteq Q$ for every j, $\langle\gamma_j\rangle$ converges to p, and $\gamma_j\cap\alpha\neq\varnothing\neq\gamma_j\cap\beta$ for every j. If Γ is a selector of arcs on E, $Q\subseteq P$, and $p \in E$, we say that *Q joins* $\Gamma(p)$ if Q joins α and β for every α, $\beta \in \Gamma(p)$. If Γ is a selector of arcs on E, we define the sets

$$\Sigma\Gamma(p) = \bigcup_{\alpha} \alpha \quad (\alpha \in \Gamma(p)), \qquad \Sigma\Gamma = \bigcup_{p} \Sigma\Gamma(p) \quad (p \in E).$$

LEMMA 1. *Let $E\subseteq P$ and let Γ be a selector of arcs on E such that the cardinality of $\Gamma(p)$ is six for every $p \in E$. Suppose there is an $\varepsilon>0$ such that* $\operatorname{dia}(\alpha)\geqq\varepsilon$ *for every $\alpha \in \Gamma(p)$, and every $p \in E$. Then there is a set $F\subseteq E$, with $E-F$ at most countable, such that for every $p \in F$ there are distinct* *arcs* $\alpha, \beta \in \Gamma(p)$, *and there is a curvilinear segment* $\gamma \subseteq \Sigma\Gamma$ *satisfying* $p \notin \gamma$, $\operatorname{dia}(\gamma) \leqq \varepsilon$, *and* $\gamma \cap \alpha \neq \varnothing \neq \gamma \cap \beta$.

PROOF. If the lemma is false, then there is an uncountable subset G of E such that for every $p \in G$ and every pair α, β of distinct arcs in $\Gamma(p)$ no such curvilinear segment exists. By covering G with a countable collection of open disks of diameters ε, it follows that for at least one such disk, say D_0, the set $G_0 = G \cap D_0$ is uncountable. For each $p \in G_0$, and each $\alpha \in \Gamma(p)$, α intersects the boundary C_0 of D_0, so that there is a terminal subarc of α which is contained in D_0 except for one end point which lies on C_0. Let $\alpha_j(p)$ $(j=1, 2, \cdots, 6)$ denote these terminal subarcs of arcs in $\Gamma(p)$ for $p \in G_0$.

Now let $\{A_j\}$ be a countable collection of open circular subarcs of C_0 which forms a basis for the usual topology on C_0. For each $p \in G_0$, by choosing the smallest indices possible, let $A_i(p)$ $(i=1, 2, \cdots, 6)$ be mutually disjoint members of $\{A_j\}$ such that $\alpha_i(p)$ intersects $A_i(p)$. Since the collection of all 6-tuples having coordinates in $\{A_j\}$ is countable, it follows that we can find distinct points $p, q \in G_0$ such that $A_i(p) = A_i(q)$ for each $i=1, 2, \cdots, 6$. We assert that this is impossible.

A trivial consequence of a point r being in G is that no two arcs in $\Gamma(p)$ intersect. Consequently the set $(D_0 - \{p\}) - \bigcup_{j=1}^{6} \alpha_j(p)$ contains six components, say D_j $(j=1, 2, \cdots, 6)$, each of which is bounded by two of the arcs $\alpha_k(p)$ $(k=1, 2, \cdots, 6)$, $\{p\}$, and a circular subarc of C_0.

There are two cases to consider. If q lies in one of the domains, say D_{j_0}, then at least four of the arcs $\alpha_k(q)$ must cross the boundary of D_{j_0}. Since these arcs are mutually disjoint, only one can cross the boundary of D_{j_0} at p. Therefore at least two of the arcs $\alpha_k(q)$ must cross one of the arcs $\alpha_j(p)$, and this contradicts $q \in G$. In the other case, q lies on one of the arcs at p, say $\alpha_{j_0}(p)$. We let D' be the union of $\alpha_{j_0}(p)$ with its two adjacent domains. Now at least two of the arcs $\alpha_k(q)$ must cross the boundary of D' at points distinct from p. No two of the arcs $\alpha_k(q)$ can cross one of the arcs $\alpha_j(p)$ since $q \in G$. Hence two of the arcs $\alpha_k(q)$ cross two of the arcs $\alpha_j(p)$. But since $q \in \Sigma\Gamma$ this implies that q, together with two of the arcs $\alpha_k(q)$, form a curvilinear segment which contradicts $p \in G$, and this completes the proof.

REMARK. It can be shown that Lemma 1 is false if the cardinality condition is replaced by a cardinality of four. On the other hand, it can be shown that Lemma 1 remains true if the cardinality condition is replaced by a cardinality of five. The merits of this stronger result, however, do not justify the added difficulties encountered in the proof.

LEMMA 2. *Let* $E \subseteq P$ *and let* Γ *be a selector of arcs on* E *such that the cardinality of* $\Gamma(p)$ *is six for every* $p \in E$. *Then there is a set* $F \subseteq E$, *with*

$E-F$ at most countable, such that for every $p \in F$ there are distinct arcs α and β in $\Gamma(p)$ such that $\Sigma\Gamma$ joins α and β.

PROOF. For each $j=1, 2, 3, \cdots$, let

$$E_j = \{p \in E: \text{dia}(\alpha) \geqq 1/j \text{ for every } \alpha \in \Gamma(p)\},$$

so that E is the union of the sets E_j. According to Lemma 1, for each $j=1, 2, 3, \cdots$, and each $k \geqq j$ there is a set $F_j(k) \subseteq E_j$, with $E_j - F_j(k)$ at most countable, and for every $p \in F_j(k)$ there is a curvilinear segment $\gamma \subseteq \Sigma\Gamma$, and distinct arcs α, $\beta \in \Gamma(p)$ such that $p \notin \gamma$, $\text{dia}(\gamma) \leqq 1/k$, and $\alpha \cap \gamma \neq \varnothing \neq \beta \cap \gamma$. We set

$$F = \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} F_j(k),$$

so that $E-F$ is at most countable. Since $\Gamma(p)$ is a finite set, it now follows that $\Sigma\Gamma$ joins at least two distinct arcs in $\Gamma(p)$ for every $p \in F$.

LEMMA 3. *Let $E \subseteq P$ and let Γ be a selector of arcs on E. Then there is a set $F \subseteq E$, with $E-F$ at most countable, and there are selectors of arcs Γ_j $(j=1, 2, \cdots, 5)$ on E such that $\Gamma(p) = \bigcup_{j=1}^{5} \Gamma_j(p)$ for every $p \in E$, and $\Sigma\Gamma$ joins $\Gamma_j(p)$ for every $p \in F$ and $j = 1, 2, \cdots, 5$.*

PROOF. Let $p \in E$, and let α, $\beta \in \Gamma(p)$. Define $\alpha \sim \beta$ if and only if $\Sigma\Gamma$ joins α and β. This defines an equivalence relation on $\Gamma(p)$. Let F be the set of all points $p \in E$ for which $\Gamma(p)$ contains less than six equivalence classes. For each $p \in E-F$ let $\alpha_j(p)$ $(j=1, 2, \cdots, 6)$ be members of six different equivalence classes of $\Gamma(p)$, and define

$$\Gamma_0(p) = \{\alpha_j(p): j = 1, 2, \cdots, 6\} \qquad (p \in E - F).$$

Then Γ_0 is a selector of arcs on $E-F$ such that $\Sigma\Gamma_0$ joins no two arcs in $\Gamma_0(p)$ for every $p \in E-F$. It follows from Lemma 2 that $E-F$ is at most countable. For each $p \in E-F$ define $\Gamma_j(p) = \Gamma(p)$ $(j=1, 2, \cdots, 5)$. For each $p \in F$ let $\Gamma_j(p)$ $(j=1, 2, \cdots, 5)$ denote the various equivalence classes in $\Gamma(p)$, where duplication is allowed in the case that $\Gamma(p)$ has less than five equivalence classes. It follows that $\Sigma\Gamma$ joins $\Gamma_j(p)$ for every $p \in F$ and every $j=1, 2, \cdots, 5$.

3. **The main result.** Let Z be an arbitrary subset of P, and let f be an arbitrary complex valued function defined on Z. We call a closed subset C of the Riemann sphere W *a missing arc cluster set of f at $p \in Z$* if $C \in \overline{\mathfrak{C}}_f(p) - \mathfrak{C}_f(p)$, where $\overline{\mathfrak{C}}_f(p)$ denotes the closure of $\mathfrak{C}_f(p)$ in $\mathfrak{H}$. If $A \subseteq Z$ we let $\bar{f}(A)$ denote the closure of $f(A)$ in W. If C is a closed, nonempty subset of W, and $\varepsilon > 0$, $C(\varepsilon)$ will denote the set of all points whose spherical distance from C does not exceed ε. It follows that if P, $Q \in \mathfrak{H}$, then $M(P, Q) \leqq \varepsilon$ if and only if $P \subseteq Q(\varepsilon)$ and $Q \subseteq P(\varepsilon)$. Several immediate

properties of M follow from this observation, and we will use them freely and without reference in the next proof.

THEOREM 1. *Let f be an arbitrary complex valued function defined on an arbitrary subset Z of the plane. Then $\mathfrak{C}_f(p)$ is a compact set in the M-topology for all but at most a countable number of points $p \in Z$.*

PROOF. Let E be the set of all points in Z at which $\mathfrak{C}_f(p)$ is not compact, and for each $p \in E$, let C_p be a missing arc cluster set of f at p. Let $\langle \varepsilon_j \rangle$ be a decreasing sequence of positive numbers which converges to zero. Then for each $j=1, 2, 3, \cdots$, and each $p \in E$, let $\alpha_j(p) \in \mathscr{A}_p(Z)$ such that

$$(1) \qquad M(C(f, p, \alpha_j(p)), C_p) \leqq \varepsilon_j/4,$$

$$(2) \qquad M(\bar{f}(\alpha_j(p)), C_p) \leqq \varepsilon_j/2.$$

For each $k=1, 2, 3, \cdots$ let $\mathfrak{G}(k, 1), \mathfrak{G}(k, 2), \cdots, \mathfrak{G}(k, n_k)$ be a covering of $\mathfrak{H}$ by compact neighborhoods such that

$$(3) \qquad \operatorname{dia}(\mathfrak{G}(k, n)) \leqq \varepsilon_k/2 \qquad (n \leqq n_k).$$

Then define

$$(4) \quad L(k, n) = \{p \in E : C_p \in \mathfrak{G}(k, n)\} \qquad (k = 1, 2, 3, \cdots; n \leqq n_k).$$

We let $\Gamma[k, n]$ be the selector of arcs on $L(k, n)$ defined by

$$(5) \quad \Gamma[k, n](p) = \{\alpha_j(p) : j = k, k+1, k+2, \cdots\} \qquad (p \in L(k, n)).$$

It follows from (2)–(5) and the properties of M that

$$(6) \qquad \bar{f}(\Sigma\Gamma[k, n]) \subseteq C_p(\varepsilon_k) \qquad (p \in L(k, n), n \leqq n_k).$$

According to Lemma 3, for each $k=1, 2, 3, \cdots$, and $n \leqq n_k$, there is a set $F(k, n) \subseteq L(k, n)$, with $L(k, n) - F(k, n)$ at most countable, and selectors of arcs $\Gamma_j[k, n]$ $(j=1, 2, \cdots, 5)$ on $L(k, n)$ such that

$$(7) \qquad \Gamma[k, n](p) = \bigcup_{j=1}^{5} \Gamma_j[k, n](p) \qquad (p \in L(k, n)),$$

and

$$(8) \quad \Sigma\Gamma[k, n] \text{ joins } \Gamma_j[k, n](p) \qquad (p \in F(k, n), j = 1, 2, \ldots, 5).$$

Define the set $F_k = \bigcup_{n=1}^{n_k} F(k, n)$, and note that each of the sets F_k contains all but at most a countable number of points in E since, for each k, $E = \bigcup_{n=1}^{n_k} L(k, n)$. Therefore the set $F = \bigcap_{k=1}^{\infty} F_k$ contains all but at most a countable number of points in E.

We will show that E is countable by showing that F is empty. Suppose to the contrary that $p \in F$. We assert that there is an $\alpha \in \mathscr{A}_p(Z)$ whose corresponding arc cluster set is C_p.

Since $p \in F$ it follows that for every $k=1, 2, 3, \cdots$, there is an $m_k \leqq n_k$ such that $p \in F(k, m_k)$. Using (8) this means that, for every $k=1, 2, 3, \cdots$, $\Sigma\Gamma[k, m_k]$ joins $\Gamma_j[k, m_k](p)$ $(j=1, 2, \cdots, 5)$. It follows from (5) and (7) that if j is an integer for which $\Gamma_j[k, m_k](p)$ is an infinite set, then there is an integer i such that

$$\Gamma_j[k, m_k](p) \cap \Gamma_i[k+1, m_{k+1}](p)$$

is an infinite set. Using this observation, we construct inductively sets $\Lambda_k = \Gamma_{j_k}[k, m_k](p)$ $(k=1, 2, 3, \cdots)$ in such a way that $\Lambda_k \cap \Lambda_{k+1}$ is an infinite set for each k. We then define inductively a sequence $\langle \alpha_k \rangle$ of arcs $\alpha_k \in \Lambda_k \cap \Lambda_{k+1}$ in such a way that $\langle \alpha_k \rangle$ is a subsequence of $\langle \alpha_k(p) \rangle$. From our construction it follows that $\Sigma\Gamma[k, m_k]$ joins α_{k-1} and α_k $(k=2, 3, 4, \cdots)$. For each $k=2, 3, 4, \cdots$, let $\langle \gamma_j(k) \rangle$ be a sequence of curvilinear segments such that

$$\gamma_j(k) \subseteq \Sigma\Gamma[k, m_k] \qquad (j = 1, 2, 3, \cdots), \tag{9}$$

$$\langle \gamma_j(k) \rangle \quad \text{converges to } p \qquad (k \text{ fixed}), \tag{10}$$

$$\gamma_j(k) \cap \alpha_{k-1} \neq \varnothing \neq \gamma_j(k) \cap \alpha_k \qquad (j = 1, 2, 3, \cdots). \tag{11}$$

It follows from (6) and (9) that

$$\bar{f}(\gamma_j(k)) \subseteq C_p(\varepsilon_k) \qquad (j, k = 1, 2, 3, \cdots). \tag{12}$$

Then using (1), (10), (11), (12), and the definition of an arc cluster set, a simple arc α at p can be constructed which passes through the sets α_k and the sets $\gamma_j(k)$, and satisfies $C(f, p, \alpha) = C_p$. This contradicts the definition of C_p and completes the proof.

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