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A TAUBERIAN GROUP ALGEBRA

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ABSTRACT. Let G be the group of real matrices

$$(x, y) = \begin{pmatrix} e^x & 0 \\ y & 1 \end{pmatrix} \qquad (x, y \in \boldsymbol{R}).$$

Every proper closed two-sided ideal of $L^1(G)$ is contained in a maximal modular two-sided ideal. The strong radical of $L^1(G)$ is the set of all $f \in L^1(G)$ with $\int f(x, y)\, dy = 0$ for almost all $x \in \boldsymbol{R}$. The strong structure spaces of $L^1(G)$ and $L^1(\boldsymbol{R})$ are homeomorphic.

Call a Banach algebra A tauberian if every proper closed two-sided ideal of A is contained in a maximal modular two-sided ideal. For completely regular Banach algebras this definition coincides with Rickart's (cf. [**2**, 2.7.25]). The L^1-group algebras of compact and of locally compact abelian groups are known to be tauberian. Probably, it is already known that the direct product of a compact and an abelian group has a tauberian group algebra. A quite different example follows.

Let G be the group of real matrices $(x, y) = \left(\begin{smallmatrix} e^x & 0 \\ y & 1 \end{smallmatrix}\right)$ $(x, y \in \boldsymbol{R})$ with its natural topology, and let $H \cong \boldsymbol{R}$ be the normal subgroup of elements $(0, y)$. The law of composition in G is

$$(x, y)(u, v) = (x + u, e^u y + v)$$

and thus $d(u, v) = du\, dv$ is the (left) Haar measure on G. Moreover,

$$(u, v)^{-1} = (-u, -e^{-u} v)$$

and

$$(u, v)^{-1}(x, y)(u, v) = (x, (1 - e^x)v + e^u y).$$

The convolution product of f and g in $L^1(G)$ is given by

$$f * g(x, y) = \int f(x + u, e^u y + v) g(-u, -e^{-u} v)\, du\, dv,$$

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and the canonical projection

$$T_H : L^1(G) \to L^1(G/H) \cong L^1(\boldsymbol{R})$$

by $(T_H f)(x) = \int f(x, y)\, dy$. Its kernel will be denoted by $K = T_H^{-1}(0)$. Let j_0 be the kernel of the trivial character of $L^1(\boldsymbol{R})$, i.e. $j_0 = \{q \in L^1(\boldsymbol{R}) \mid \hat{q}(0) = \int q(y)\, dy = 0\}$. Now K can easily be identified with $L^1(\boldsymbol{R}, j_0)$, which in turn contains $L^1(\boldsymbol{R}) \times j_0$ as a total subset. In other words, the elements of K can be approximated by finite sums of elements $p \otimes q$—$p \otimes q(x, y) = p(x)q(y)$—with p and q from $L^1(\boldsymbol{R})$ and $\hat{q}(0) = \int q(y)\, dy = 0$.

The element $(-\log z, 0)$ for $z > 0$ defines the inner automorphism i_z of G,

$$i_z(x, y) = (x, z^{-1}y),$$

and i_z induces an isometric automorphism M_z of $L^1(G)$ given by

$$(M_z f)(x, y) = z^{-1} \cdot f(x, z^{-1}y) \quad \text{for } f \in L^1(G).$$

Similarly,

$$(M_z f)(y) = z^{-1} f(z^{-1}y)$$

defines an isometric automorphism of $L^1(\boldsymbol{R})$. By [**1**, Chapter 1, §2.2, p. 6],

$$\lim_{z \to 0} (M_z f) * g = \hat{f}(0) g \quad \text{for } f, g \in L^1(\boldsymbol{R}). \tag{1}$$

REMARK 1. This property is essential to the main proof and seems to have no analogue in general locally compact groups.

$$\lim_{z \to 0} (M_z f) * g = 0 \quad \text{for } f \in K,\ g \in L^1(G). \tag{2}$$

PROOF. It suffices to prove (2) for $f = p \otimes q$ with $p, q \in L^1(\boldsymbol{R})$, $\hat{q}(0) = 0$ and $g = h \otimes k$ with h, k continuous functions with compact support, since these functions are total in K and $L^1(G)$ respectively. Now $M_z f = p \otimes M_z q$ and

$$\begin{aligned}(M_z f * g)(x, y) &= \int p(x + u)(M_z q)(e^u y + v) h(-u) k(-e^{-u} v)\, du\, dv \\ &= \int p(x + u) h(-u) \int e^u (M_z q)(e^u (y + v)) k(-v)\, dv\, du \\ &= \int p(x + u) h(-u) ((M_{e^{-u} z} q) * k)(y)\, du.\end{aligned}$$

Consequently

$$|M_z f * g|_1 \leq \int |p(x + u)|\, |h(-u)|\, |(M_{e^{-u} z} q) * k|_1\, dx\, du.$$

The support of h is contained in an interval $[-a, +a]$ and, by (1), $|M_t q * k|_1 \leq \varepsilon$ for $t \leq \delta = \delta(\varepsilon)$. Now $e^{-u} z \leq \delta$ for $|u| \leq a$ and $z \leq e^{-a}\delta$; hence we obtain

$$|M_z f * g|_1 \leqq |p|_1 \int_{-a}^{+a} |h(-u)|\, |(M_{e^{-u}z} q) * k|_1\, du$$
$$\leqq \varepsilon\, |p|_1 \cdot |h|_1 \quad \text{for } z \leqq e^{-a}\delta.$$

REMARK 2. Let $f=p\otimes q$ with p, q in $L^1(\boldsymbol{R})$, $q\geqq 0$ and $\hat{q}(0)=1$, and let $g=h\otimes k$ be as above. A similar computation leads to

$$|M_z f * g - g|_1 \leqq |p|_1\, |h|_1 \sup_{|u|\leqq a} |(M_{e^{-u}z} q) * k - k|_1 + |p * h - h|_1 \cdot |k|_1.$$

Again, the supremum can be made arbitrarily small, and if p is chosen conveniently from an approximate identity $\{p_i\}_i$ the second term becomes small. Also $|M_z f|_1 = |p_i|_1 \cdot \hat{q}(0)=1$. Thus a density argument for the g's shows that $\{p_i\otimes M_z q\}_{(i,z)}$ is an approximate identity for $L^1(G)$ if $(i,z)\geqq (j,z')$ is defined to mean $i\geqq j$ and $z\leqq z'$.

THEOREM. *If J is a proper closed two-sided ideal in $L^1(G)$, then so is $J+K$.*

PROOF. By §4.6(ii), Chapter 8 in **[1]** $J+K$ is a closed, two-sided ideal. We will show that if $J+K=L^1(G)$ then $J=L^1(G)$. Let $f\in L^1(G)$ and let $\{p_i\otimes M_z q\}_{(i,z)}$ be the approximate identity described in Remark 2. Set $q_i=p_i\otimes q$ and $q_{i,z}=p_i\otimes M_z q = M_z q_i$. Since $J+K=L^1(G)$, $q_i=q_i'+q_i''$ with $q_i'\in J$, $q_i''\in K$. Since J and K are invariant under left translations L_g, $(L_g f)(g')=f(g^{-1}g')$ and right translations R_g, $(R_g f)(g')=\Delta(g)f(g'g)$, they are invariant under $M_z=L_g R_g$, with $g=(-\log z, 0)$, and it follows that $q_{i,z}=q_{i,z}'+q_{i,z}''$ with $q_{i,z}'=M_z q_i'\in J$, $q_{i,z}''=M_z q_i''\in K$. For $\varepsilon>0$ there exists (i_0, z_0) such that $|f-q_{i,z} * f|_1<\varepsilon$ provided $(i,z)\geqq(i_0,z_0)$, i.e. provided $i\geqq i_0$ and $z\leqq z_0$. Consequently,

$$|f - q_{i,z}' * f|_1 \leqq \varepsilon + |q_{i,z}'' * f|_1.$$

Since $q_i''\in K$, (2) implies

$$\lim_{z\to 0} |q_{i,z}'' * f|_1 = \lim_{z\to 0} |M_z q_i'' * f|_1 = 0,$$

i.e.

$$|q_{i,z}'' * f|_1 \leqq \varepsilon \quad \text{for } z \leqq \delta(i,\varepsilon) \leqq z_0.$$

Hence

$$|f - q_{i,z}' * f|_1 \leqq 2\varepsilon \quad \text{for } i \geqq i_0,\ z \leqq \delta(i,\varepsilon).$$

Since ε was arbitrary, this implies $f\in J$.

COROLLARY 1. *The maximal modular two-sided ideals of $L^1(G)$ contain K.*

PROOF. If J is a maximal modular two-sided ideal in $L^1(G)$, $J+K$ is a proper modular two-sided ideal containing J. By maximality $J=J+K\supset K$.

REMARK 3. Not all closed two-sided ideals of $L^1(G)$ contain K: Let j_+ $[j_-]$ be the ideal in $L^1(\boldsymbol{R})$ which consists of the functions g whose Fourier transforms $\hat{g}$ vanish for all positive [negative] λ, i.e.

$$\hat{g}(\lambda) = \int e^{i\lambda y} g(y)\, dy = 0 \qquad (\lambda > 0\ [\lambda < 0]).$$

By continuity $\hat{g}(0)=0$; hence $j_\pm \subset j_0$. Let J_+ be the set of $f \in L^1(G)$ with $(y \to f(x,y)) \in j_+$ for almost all x. Let J_- be similarly defined. By the uniqueness of the Fourier transform $j_+ \cap j_- = \{0\}$, hence also $J_+ \cap J_- = \{0\}$. $J_+ \oplus J_- \subset K$ and neither J_+ nor J_- contains K. That J_+ and J_- are closed two-sided ideals can be seen as follows: The operator A_λ defined by $(A_\lambda f)(x,y) = e^{i\lambda y} f(x,y)$ is an isometry of $L^1(G)$ and thus maps closed subspaces into closed subspaces. In particular $K_\lambda = A_\lambda^{-1} K$ and $J_+ = \bigcap_{\lambda > 0} K_\lambda$ are closed. It is easy to check that K_λ is invariant under left-translations and that the right-translation $R_{(a,b)}$ maps K_λ onto $K_{\lambda e^{-a}}$. Hence, J_+ is a two-sided ideal.

COROLLARY 2. *T_H induces a homeomorphism of the strong structure spaces of $L^1(G)$ and $L^1(\boldsymbol{R})$ and K, the kernel of T_H, is the strong radical of $L^1(G)$.*

PROOF. By §4.4, Chapter 3 of [**1**] the isomorphism $L^1(G/H) \cong L^1(G)/K$ is algebraic and isometric. By Theorem 2.6.6 of [**2**] and Corollary 1 the strong structure spaces of $L^1(\boldsymbol{R}) \cong L^1(G/H)$ and $L^1(G)$ are homeomorphic. Since $L^1(\boldsymbol{R})$ is strongly semisimple its strong radical is $\{0\}$, and the inverse image K of $\{0\}$ under T_H is the strong radical of $L^1(G)$.

COROLLARY 3. *$L^1(G)$ is a completely regular tauberian Banach algebra.*

PROOF. By the theorem, $T_H(J) \cong (J+K)/K$ is a proper closed two-sided ideal iff J is one. Since every such ideal $T_H(J)$ is contained in a maximal modular two-sided ideal M of $L^1(\boldsymbol{R})$, J is contained in $T_H^{-1}(M)$ which is itself modular. Thus $L^1(G)$ is tauberian (cf. 2.7.25 of [**2**]). Since $L^1(\boldsymbol{R})$ is completely regular 2.7.4 of [**2**] implies that $L^1(G)$ is completely regular.

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