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REGULAR MATRICES AND P-SETS IN $\beta N \backslash N$

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ABSTRACT. A P-set is a closed set which is interior to any zero set (closed G_δ) which contains it. Henriksen and Isbell showed that the 'support set' in $\beta N \backslash N$ of a nonnegative regular matrix is a P-set. We show that each such support set contains a family of 2^c pairwise disjoint perfect nowhere dense P-sets, so that not every P-set comes from a matrix. Moreover, each of the P-sets produced is the support of a Borel probability measure on $\beta N \backslash N$.

1. **Introduction.** Let $T=(t_{mn})$ be a nonnegative regular matrix, F_T the filter of subsets A of the positive integers N such that $T-\lim \chi_A=1$, and K_T the corresponding closed set in $\beta N \backslash N$. Henriksen and Isbell **[H-I]** showed that K_T is perfect and a 'P-set', i.e., is a closed set which is interior to any closed G_δ set which contains it. It is natural to ask whether every such subset of $\beta N \backslash N$ is related to a nonnegative matrix as above. This question is resolved by

THEOREM. *Assume the continuum hypothesis. Then there exists a family of 2^c pairwise disjoint perfect nowhere dense P-sets contained in K_T. Moreover each of these P-sets is the support set of a Borel probability measure on $\beta N \backslash N$.*

According to **[H-I]** the set K_T has nonvoid intersection with each member of an uncountable family of pairwise disjoint clopen subsets of $\beta N \backslash N$, so it cannot be the support set of a Borel measure. Hence

COROLLARY. *Under the continuum hypothesis there exist perfect nowhere dense P-sets in $\beta N \backslash N$ which do not correspond to any regular matrix.*

In the fourth section we prove a more general version of the main theorem. Namely, let $T=(t_{mn})$ satisfy

(1) $\lim(m \to \infty)\, t_{mn}=0$ for all n,

(2) $\sup_m \sum_n |t_{mn}| < \infty$,

(3) $\limsup(m \to \infty) \sum_n |t_{mn}| > 0$.

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Then the support set K_T of T can again be defined (although not in so simple a fashion as above). As far as the author knows, unless T is nonnegative it is not known whether K_T is a P-set. Nevertheless we show that K_T contains a family of 2^c pairwise disjoint perfect nowhere dense P-sets, each the support of a Borel probability measure.

2. **Preliminaries.**

2.1. NOTATION. If $A \subset N$, let A' be its closure in βN and $A^* = A' \cap \beta N \backslash N$. Then $N^* = \beta N \backslash N$. Note that $K_T = \bigcap \{A^* : A \in F_T\}$. We write cl R for the closure of any subset R of N^*. $C^*(N)$ is the space of bounded real functions on N. If $f \in C^*(N)$, f' is its extension to βN, and f^* the restriction of f' to N^*.

$T = (t_{mn})$ will always (except in §4) be a nonnegative regular matrix. If c_0 is the space of real functions on N which vanish at infinity, then $T(c_0) \subset c_0$, so T induces an operator T^* on $C(N^*)$ by the formula $T^*f^* = (Tf)^*$ $(f \in C^*(N))$. By regularity of T, $T^*1 = 1$. If $p \in N^*$, let m_p be the Borel probability measure representing the functional $f \to T^*f(p)$ $(f \in C(N^*))$, K_p the support set of m_p, and $K = \text{cl} \bigcup \{K_p : p \in N^*\}$. If T is nonnegative, then it is easy to see that $K = K_T$, where K_T is as in the Introduction (see [A]).

2.2. DEFINITION. Let $T = (t_{mn})$. $S = (s_{mn})$ is a *submatrix* of T if there is a sequence $m(k)$ of integers such that the kth row of S is the $m(k)$th row of T.

2.3. PROPOSITION. *Let W be a clopen subset of N^*. Then* cl $\bigcup \{K_p : p \in W\}$ *is the support set of a submatrix of T, and hence is a P-set.*

PROOF. Let $A \subset N$ with $A^* = W$. Let $\varphi : N \to A$ be an order preserving bijection, and $\varphi^* : N^* \to W$ be its extension to N^* (again a bijection). Let S be the matrix such that $Sf(n) = f(\varphi n)$ $(f \in C^*(N))$, and $R = S \circ T$. Then $Rf(n) = Tf(\varphi n)$, so R is a submatrix of T. On $C(N^*)$, $R^* = S^* \circ T^*$, and if $p \in N^*$, $f \in C(N^*)$,

$$R^*f(p) = T^*f(\varphi^*p) = \int_{K_q} f\, dm_q,$$

where $q = \varphi^*p$. Hence the support of the matrix R is

$$K_R = \text{cl} \bigcup \{K_q : q = \varphi^*p, p \in N^*\}.$$

But since $\varphi^* : N^* \to W$ is a bijection, $K_R = \text{cl} \bigcup \{K_p : p \in W\}$, and by the Henriksen-Isbell theorem K_R is a P-set.

2.4. PROPOSITION. *If J is a P-set, then $L = \text{cl} \bigcup \{K_p : p \in J\}$ is a P-set.*

PROOF. Let $f \in C(N^*)$ be nonnegative with $L \subset f^{-1}(0)$. We must show there is an open V with $L \subset V \subset f^{-1}(0)$. Now if $p \in J$, $Tf(p) = 0$ (since f

vanishes on K_p), so $J \subset (Tf)^{-1}(0)$. But J is a P-set, so there exists open U with $J \subset U \subset (Tf)^{-1}(0)$. By compactness of J there exists clopen W with $J \subset W \subset U$. Now $p \in W$ implies $Tf(p)=0$, i.e., the integral of f with respect to the measure m_p is 0. Since f is nonnegative, $K_p \subset f^{-1}(0)$. Hence

$$L \subset \mathrm{cl} \bigcup \{K_p : p \in W\} \subset f^{-1}(0).$$

Since W is clopen, 2.3 implies that the set in the middle is a P-set, so there exists open V with $L \subset V \subset f^{-1}(0)$.

2.5. COROLLARY. If p is a P-point, then K_p is a P-set.

3. **Proof of the theorem.** We assume $T=(t_{mn})$ satisfies

$$\lim(m \to \infty) \sup\{t_{mn} : n \in N\} = 0.$$

(If this condition is not satisfied, then K_T contains a clopen set W **[Ra]**, and we can construct a matrix which does satisfy the condition, whose support set is contained in W.) By Lemma 4.1.2. of **[P]** we may assume without loss of generality that T is *truncated*, i.e., there exist sequences $\{r(m)\}$ and $\{s(m)\}$, both increasing monotonically to infinity, with $t_{mn}=0$ whenever $n<r(m)$ or $n>s(m)$. By regularity of T we may also assume each row sum is 1. Taking T to have this form, it follows that there exist $m(1)<m(2)<\cdots$ such that the corresponding rows have disjoint supports, i.e., if $i \neq j$, then $t_{m(i)k}>0$ implies $t_{m(j)k}=0$. Let S be the matrix having as its kth row the $m(k)$th row of T. Then $K_S \subset K_T$, for if $A \subset N$ and $T-\lim \chi_A=1$, then $S-\lim \chi_A=1$.

Now S is a nonnegative regular matrix with row sums$=1$, and whose rows have disjoint supports. If $p \in N^*$, let L_p be the support of the measure representing the functional $f \to (S^*f)(p)$ $(f \in C(N^*))$. Assuming the continuum hypothesis, there are 2^c P-points in N^* **[R]**, and if p is a P-point, then 2.5 implies L_p is a P-set. Each L_p is nowhere dense, since the support of a Borel measure in N^* is nowhere dense. (Every clopen subset of N^* contains a family of c pairwise disjoint clopen sets.) Lemma 3.1 below will imply that if p and q are distinct, then L_p and L_q are disjoint. To show that if p is a P-point, L_p is perfect, note that L_p is a P-set. It is easy to see that an isolated point in a P-set must be a P-point. We show in Lemma 3.2 that K_S contains no P-points.

3.1. LEMMA. *Let $S=(s_{mn})$ be a nonnegative regular matrix such that each row sum is* 1, *and distinct rows have disjoint supports. Let L_p $(p \in N^*)$ be as above. If $p \neq q$, then L_p and L_q are disjoint.*

PROOF. First we show that S maps the set $\{f \in C^*(N) : 0 \leqq f \leqq 1\}$ on itself, and hence S^* maps $\{f \in C(N^*) : 0 \leqq f \leqq 1\}$ on itself. If $0 \leqq f \leqq 1$, define $g \in C^*(N)$ to have the value $f(m)$ for each k such that $s_{mk} \neq 0$, and let $g(k)=0$ if $s_{mk}=0$ for all m. Then $Sg(m)=\sum_k s_{mk} g(k)=f(m)$ for all m.

Now let p and q be distinct points of N^*, and choose $f \in C(N^*)$ with $0 \leqq f \leqq 1, f(p)=1, f(q)=0$. Choose $g \in C(N^*)$ with $0 \leqq g \leqq 1$, $S^*g=f$. Since $S^*g(q)=0$, g vanishes on L_q. If $L_q \cap L_p \neq \varnothing$, then $f(p)=S^*g(p)<1$, a contradiction. Hence L_p and L_q are disjoint.

3.2. LEMMA. *Let $S=(s_{mn})$ be a nonnegative regular matrix such that*

(a) $\lim(m \to \infty) \sup\{s_{mn}: n \in N\}=0$, *and*

(b) *the rows of S have disjoint supports.*

Then K_S contains no P-points.

PROOF. First we show that if F is any ultrafilter in N and t is the infimum over $A \in F$ of the quantities $\limsup(m \to \infty) \sum \{s_{mn}: n \in A\}$, then $t=0$. Suppose $t>0$. By (a) we may assume $s_{mn}<t/4$ for all m and n. Choose $A \in F$ such that

$$\limsup(m \to \infty) \sum \{s_{mn}: n \in A\} < (\tfrac{4}{3})t.$$

Then M exists such that $\sum \{s_{mn}: n \in A\}<(\frac{4}{3})t$ whenever $m \geqq M$. Let $L_m=\{k: s_{mk} \neq 0\}$, so that, by (b), L_m and L_p are disjoint whenever $m \neq p$. For each m, let $g(m)$ be the largest integer in L_m such that

$$\sum \{s_{mn}: n \in A, n<g(m)\}<(\tfrac{3}{4})t.$$

Then, unless $\sum \{s_{mn}: n \in A\}<(\frac{3}{4})t$, we have

$$\sum \{s_{mn}: n \in A, n \leqq g(m)\} \geqq (\tfrac{3}{4})t.$$

Let $B=\bigcup_m L_m \cap [0, g(m)) \cap A$. Since F is an ultrafilter, either $B \in F$ or $C=A \backslash B \in F$. But $B \notin F$ because

$$\limsup(m \to \infty) \sum \{s_{mn}: n \in B\} \leqq (\tfrac{3}{4})t<t.$$

We shall obtain a contradiction by showing $C \notin F$ as well. Let $m \geqq M$. If $\sum \{s_{mn}: n \in A\}<(\frac{3}{4})t$, then $\sum \{s_{mn}: n \in C\}<(\frac{3}{4})t<(\frac{5}{6})t$. If $\sum \{s_{mn}: n \in A\} \geqq (\frac{3}{4})t$, then since $s_{m, g(m)}<t/4$ we have

$$\begin{aligned}
(\tfrac{4}{3})t &> \sum \{s_{mn}: n \in A\} \\
&= \sum \{s_{mn}: n \in A, n \leqq g(m)\}+\sum \{s_{mn}: n \in C \backslash\{g(m)\}\} \\
&\geqq (\tfrac{3}{4})t+\sum \{s_{mn}: n \in C \backslash\{g(m)\}\} \\
&> t/2+s_{m, g(m)}+\sum \{s_{mn}: n \in C \backslash\{g(m)\}\} \\
&= t/2+\sum \{s_{mn}: n \in C\},
\end{aligned}$$

whence again $\sum \{s_{mn}: n \in C\}<(\frac{4}{3})t-t/2=(\frac{5}{6})t$. Hence

$$\limsup\{m \to \infty) \sum \{s_{mn}: n \in C\} \leqq (\tfrac{5}{6})t<t,$$

so $C \notin F$.

Now suppose F is the filter of sets corresponding to a P-point p in N^*. By what we have just shown, there exists, for each n, an $A(n) \in F$ with

$$\limsup(m \to \infty) \sum \{s_{mk}: k \in A(n)\} < 1/n.$$

Since p is a P-point, there exists $A \in F$ with $A \backslash A(n)$ finite for all n, whence

$$\limsup(m \to \infty) \sum \{s_{mk}: k \in A\} = 0.$$

If $B = N \backslash A$, then $\lim(m \to \infty) \sum \{s_{mk}: k \in B\} = 1$. Hence $B \in F_S$, and $K_S \subset B^*$ while $p \notin B^*$.

4. **A more general result.** As pointed out by the referee, it is not necessary to assume that $T = (t_{mn})$ is regular and nonnegative, but only that it satisfy conditions (1), (2) and (3) of the Introduction. Then $T(c_0) \subset c_0$, and again we get an operator T^* on $C(N^*)$, with support defined by the formula $K_T = \mathrm{cl} \bigcup \{K_p : p \in N^*\}$. To the author's knowledge, unless T is nonnegative it is not known whether K_T is a P-set (see **[H-I]** and the proofs of the Henriksen-Isbell theorem which occur on p. 440 of [**A**] and p. 414 of **[H-S]**. Apparently the difficulty is that unless $T \geqq 0$, it is not clear how to describe K_T, as in the first sentence of the Introduction, as the intersection of summable sets). To show the theorem holds for this case, we show that K_T contains the support set of a nonnegative regular matrix.

As in the first paragraph of §3, we may assume T is truncated, and choose a submatrix $S = (s_{mn})$ of T such that distinct rows are disjoint, and such that (1), (2), and (3) are satisfied. Then (as can be seen from the proof of 2.3) $K_S \subset K_T$. For each m and n, let $p_{mn} = \max\{s_{mn}, 0\}$, $q_{mn} = -\min\{s_{mn}, 0\}$, $P = (p_{mn})$, and $Q = (q_{mn})$. P and Q both satisfy (1) and (2), and at least one of them (say P) satisfies (3). By taking a submatrix of P if need be, we may assume

$$\liminf(m \to \infty) \sum_n p_{mn} > 0. \tag{4}$$

Let $P_m = \sum_n p_{mn}$, $r_{mn} = p_{mn}/P_m$, and $R = (r_{mn})$. Then $K_R \subset K_T$, and R is regular and nonnegative.

4.1. Question. It is not known if the continuum hypothesis is needed to prove the existence of P-points in N^*. Is it needed to prove the existence of Borel measures on N^* whose support sets are P-sets?

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