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INVARIANT SUBSPACES OF INFINITE CODIMENSION FOR SOME NONNORMAL OPERATORS

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ABSTRACT. Let $\varphi \in C'[-1, 1]$. For $f \in L^2(-1, 1)$ define

$$T_\varphi f(s) = sf(s) + \frac{\varphi(s)}{\pi} \int_{-1}^{1*} \frac{\bar{\varphi} f(t)}{s - t} \, dt.$$

Our main result says T_φ has invariant subspaces of infinite codimension.

Introduction. For $\varphi \in C'[-1, 1]$, consider the singular integral operator defined on $f \in L^2(-1, 1)$ by

$$(1) \qquad T_\varphi f(s) = sf(s) + \frac{\varphi(s)}{\pi} \int_{-1}^{1*} \frac{\bar{\varphi} f(t)}{s - t} \, dt, \quad \text{a.e. } s \in [-1, 1].$$

If the singular integral is interpreted as a Cauchy principal value then T_φ defines a bounded operator on $L^2(-1, 1)$. It is known that the operators T_φ have invariant subspaces arising from eigenvalues of T_φ^*. It will be a corollary to our main result (Theorem 1) that T_φ has invariant subspaces of infinite codimension.

1. An operator A on a Hilbert space $\mathscr{H}$ is said to be hyponormal in case its self-adjoint self-commutator $D = A^*A - AA^*$ is positive semidefinite ($D \geqq 0$). An operator A will be called completely nonnormal in case there are no subspaces reducing the operator on which the operator is normal.

The operators T_φ defined by (1) are hyponormal. Indeed $T_\varphi^* T_\varphi - T_\varphi T_\varphi^* = (2/\pi)\langle\ , \varphi\rangle\varphi$. In the cases where $\varphi(t) \neq 0$ a.e. (which we will assume from now on) the operator T_φ is completely nonnormal.

A point $\lambda \in C$ will be called a bare point of a set $F \subset C$ in case there is a circle C_λ such that $C_\lambda \cap F = \{\lambda\}$ and $(C_\lambda)^\circ$ (for $E \subset C$, E° denotes the interior of E) is contained in the complement of F. We will denote the spectrum of an operator A by $\mathrm{sp}(A)$. The number $r_{\mathrm{sp}}(A) = \sup\{|\lambda| : \lambda \in \mathrm{sp}(A)\}$ is called the spectral radius of the operator A. An operator A is said to be normaloid

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in case $r_{sp}(A)=\|A\|$. Hyponormal operators are normaloid (Stampfli [**9**, Theorem 1]).

A sequence $\{\lambda_n\}_{n=1}^{\infty}$ of points in the unit disc is called a Blaschke sequence in case $\sum_{n=1}^{\infty} 1-|\lambda_n|<\infty$. A subset of the unit disc is a zero set for a nonzero bounded analytic function if and only if it is a Blaschke sequence (see, e.g. Rudin [**6**, p. 302]).

We will also use the following result due to von Neumann (for a proof, see Halmos [**3**, Problem 180]).

THEOREM vN. *If p is any polynomial and A an operator of norm at most 1, then* $\|p(A)\| \leqq \max\{|p(\lambda)|:|\lambda|\leqq 1\}$.

The argument used in the following theorem is similar to the argument used in Shields and Wallen [**8**, Lemma 5].

THEOREM 1. *Assume A is completely nonnormal and hyponormal on $\mathscr{H}$. Suppose $\{\bar{\lambda}_n\}_{n=1}^{\infty}$ is a sequence of eigenvalues of A^* with eigenvectors g_n such that $\lambda_n \to \lambda_0$, where λ_0 is a bare point of* $\operatorname{sp}(A)$. *Then for some infinite subsequence* $\operatorname{span}(g_{n_k})_{k=1}^{\infty} \neq \mathscr{H}$.

PROOF. Let $C_{\lambda_0}=\{\lambda:|\lambda-\mu_0|=r\}$ be a circle such that $C_{\lambda_0}\cap \operatorname{sp}(A)=\{\lambda_0\}$ and $C_{\lambda_0}^{\circ} \subset C\backslash \operatorname{sp}(A)$. Consider the operator

$$A_1 = [(A-\mu_0)^{-1} - \mu_0 I]/\|(A-\mu_0)^{-1} - \mu_0 I\|.$$

Clearly $\|A_1\|=1$ and $\bar{\mu}_n=[(\bar{\lambda}_n-\bar{\mu}_0)^{-1}-\bar{\mu}_0]/\|(A-\mu_0)^{-1}-\mu_0 I\|$ is a sequence of eigenvalues of A_1^*. One sees easily that $|\mu_n|\to r_{sp}(A_1)=\|A_1\|=1$. The identity $r_{sp}(A_1)=\|A_1\|$ holds since inverses of hyponormal operators are hyponormal (see Stampfli [**10**, Lemma 1]). Pick a subsequence $\{\mu_{n_k}\}_{k=1}^{\infty}$ of the μ_n's such that $\sum_{k=1}^{\infty}[1-|\mu_{n_k}|]<\infty$. Let $B(z)$ be a bounded analytic function in the unit disc with $B(z)=0$ if and only if $z=\mu_{n_k}$ for some $k\geqq 2$. The Fejér means $P_n(z)$ of the sequence of nth partial sums of the power series expansion of $B(z)$ form a bounded sequence of polynomials converging uniformly on compact subsets of the disc to $B(z)$. Choose a $g\in\mathscr{H}$ such that $(g, g_{n_1})\neq 0$. Since $\|A_1\|\leqq 1$, by Theorem vN, $P_k(A_1)g$ is a bounded sequence in $\mathscr{H}$. We select a weakly convergent subsequence $P_{k_j}(A_1)g \rightharpoonup g'$. Then

$$(g', g_{n_p}) = \operatorname*{Lim}_j (P_{k_j}(A_1)g, g_{n_p}) = \operatorname*{Lim}_j P_{k_j}(\mu_{n_p})(g, g_{n_p}) = B(\mu_{n_p})(g, g_{n_p}).$$

It follows that $g'\neq 0$ and $g' \perp g_{n_k}$ for $k\geqq 2$. This completes the proof.

It should be remarked that Theorem 1 is true, for example, if $(aT+b)/(cT+d)$ is normaloid whenever $(aT+b)/(cT+d)$ is bounded.

2. In this section we will describe the spectrum of the operators T_φ for $\varphi\in C'[-1,1]$. It will then be clear that T_φ has invariant subspaces of

infinite codimension. Actually the spectrum of T_φ for $\varphi \in C'[-1, 1]$ was described by Putnam [4]. We will give a slightly improved description.

A completely hyponormal operator has no eigenvalues. Indeed $A^*A \geqq AA^*$ says $\|(A-\lambda I)x\|^2 \geqq \|A^*-\bar{\lambda}I)x\|^2$ for all $\lambda \in C$. The next lemma establishes that eigenvalues of T_φ^* have unit multiplicity.

LEMMA 1. *Suppose A is a completely nonnormal operator such that $A^*A-AA^*=\langle\ , \varphi\rangle\varphi$. Then* dim ker $A^* \leqq 1$.

PROOF. Since $A^*A \geqq AA^*$, ker $A=(0)$. Suppose now f_1, f_2 are two nonzero elements of ker A^*. Then $A^*Af_i=\langle f_i, \varphi\rangle\varphi$. Since ker $A=(0)$ it follows that $\langle f_i, \varphi\rangle \neq 0$, $i=1, 2$. Therefore, $A^*A[f_1-(\langle f_1, \varphi\rangle/\langle f_2, \varphi\rangle)f_2]=0$. Again since ker $A=(0)$ we must have $\langle f_2, \varphi\rangle f_1=\langle f_1, \varphi\rangle f_2$ and this completes the proof.

Recall that an operator T is Fredholm in case the range of T is closed and both ker A and ker A^* are finite dimensional. In case T is Fredholm we define the index of T by $i(T)=\dim \ker T-\dim \ker T^*$.

The next lemma was pointed out to the author by D. N. Clark.

LEMMA 2. *Assume that for λ in an open set Ω, $A-\bar{\lambda}I$ is Fredholm, one-to-one and $i(A-\bar{\lambda}I)=-1$. Then the eigenfunctions f_λ satisfying $[A^*-\lambda I]f_\lambda=0$ are analytic in λ.*

PROOF. Let $\lambda_0 \in \Omega$. The hypotheses imply that $A^*-\lambda_0$ has a right inverse R_{λ_0} satisfying $(A^*-\lambda_0 I)R_{\lambda_0}=I$. It is easy to see that $f_\lambda=[A^*-\lambda_0]R_\lambda f_{\lambda_0}$, for $\lambda \in \Omega$. Analyticity follows since, for $|\lambda-\lambda_0|<\|R_{\lambda_0}\|^{-1}$, R_λ has the form $R_\lambda=R_{\lambda_0}\sum_{n=0}^{\infty}(\lambda-\lambda_0)^n R_{\lambda_0}^n$.

Let $\mathscr{K}$ denote the ideal of compact operators acting on $\mathscr{H}$. The essential spectrum of an operator $A \in \mathscr{B}(\mathscr{H})$ is the spectrum of the coset determined by A in the Calkin algebra $\mathscr{B}(\mathscr{H})/\mathscr{K}$. The essential spectrum is the set of complex λ such that $A-\lambda$ is not Fredholm (Schwartz [7, Lemma 1]).

Schwartz [7, Theorem 4] has shown that the essential spectrum of T_φ is the boundary of the curvilinear rectangle $R_\varphi=\{z=x+iy: -|\varphi(x)|^2 \leqq y \leqq |\varphi(x)|^2, -1 \leqq x \leqq 1\}$. Now it is known that the spectrum of a nonnormal hyponormal operator T must have positive Lebesgue planar measure. This result is due to Putnam [5] and when T^*T-TT^* is compact it is due to Clancey [2]. Using Lemmas 1 and 2 and the above remarks we can conclude:

THEOREM 2. *The spectrum of T_φ is R_φ. The boundary of R_φ is the essential spectrum of T_φ. Each point $\lambda \in R_\varphi^\circ$ is an eigenvalue of T_φ^* of unit multiplicity. If f_λ denotes the eigenfunction of T_φ^* corresponding to $\lambda \in R_\varphi^\circ$ then f_λ is an L^2-valued analytic function on R_φ°.*

It is now obvious from Theorems 1 and 2 that whenever λ_n is a sequence in R_φ°, $\lambda_n \to \lambda$ where λ is boundary point of R_φ, then some infinite subsequence of the eigenfunctions f_n of T_φ^* corresponding to λ_n fails to span L^2.

It should be remarked that one can compute the eigenfunctions of T_φ^* explicitly. See, for example, Tricomi [**11**, Chapter 4, §4]. An interesting problem is to prove or disprove that the set of all eigenfunctions f_λ of T_φ^* for $\lambda \in R_\varphi^\circ$ span $L^2(-1, 1)$.

In the special case $\varphi(t)=(1-t^2)^{1/4}$ the operator T_φ is the unilateral shift (see Clancey [**1**]). This is the only operator of the form T_φ where the complete structure of the invariant subspaces is known.

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