PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 2, February 1973

SIMILARITY OF MATRICES OVER FINITE RINGS

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ABSTRACT. It is shown that questions of similarity of certain invertible matrices over a finite ring can be reduced to questions of similarity over finite fields through the application of canonical epimorphisms.

Suprunenko has shown in [3] that two invertible matrices over Z/Z_m whose orders are relatively prime to m are similar if and only if their canonical images are similar over the fields Z/Zp for each prime divisor p of m. An analogous result holds for invertible matrices over any finite commutative ring with identity.

Preliminaries. If R is a finite commutative ring with identity, then R is uniquely a ring direct product of finite local rings [**1**, Theorem 8.7, p. 90]. Suppose that $R=\prod_{i=1}^{t} R_i$, where R_i is a finite local ring with maximal ideal M_i. Each R_i has cardinality $p_i^{e_i}$ for some prime p and has associated with it a canonical projection,

$$h_i: R_i \to R_i/M_i = \mathrm{GF}(p_i^{f_i}).$$

Setting $k_i=\mathrm{GF}(p_i^{f_i})$ we will say that the finite fields $\{k_i: i=1, 2, \cdots, t\}$ are the *fields associated with* R.

Observe that the decomposition of R carries over to the general linear group of degree n over R yielding $\mathrm{GL}_n(R)\cong\prod_{i=1}^{t}\mathrm{GL}_n(R_i)$. Furthermore, for each i, the projection h_i induces an epimorphism,

$$h_i: \mathrm{GL}_n(R_i) \to \mathrm{GL}_n(k_i).$$

If $\mathrm{GL}_n(R_i)$ is taken as the group of n by n invertible matrices over R_i, then h_i is simply reduction modulo M_i. Note that the kernel of h_i, K_i, has cardinality a power of p_i and thus is a solvable group.

The following corollary to P. Hall's extension of the Sylow theorems [**2**, Theorem 9.3.1, p. 141] is the key result needed for Theorems 1 and 2.

Observation. Let G be a finite group with solvable normal subgroup K and let $\bar{G}=G/K=\{\bar{g}|g \in G\}$. Let α and β be elements of G with $(|\alpha|, |k|)=1=(|\beta|, |K|)$. Then $\bar{\alpha}\sim\bar{\beta}$ implies $\alpha\sim\beta$.

Received by the editors April 12, 1972.

AMS (MOS) subject classifications (1970). Primary 13H99, 15A33, 15A21, 20D20, 20H25.

Key words and phrases. Similarity, finite local ring, finite solvable group.

PROOF. Since $\bar{\alpha}=\bar{\gamma}^{-1}\bar{\beta}\bar{\gamma}$ for some γ it follows that $\langle\alpha\rangle K=\langle\gamma^{-1}\beta\gamma\rangle K$. By P. Hall's theorem it follows that $\langle\alpha\rangle$ and $\langle\gamma^{-1}\beta\gamma\rangle$ are conjugate in $\langle\alpha\rangle K$. Thus there is a μ in K and $r>0$ such that $\mu^{-1}\gamma^{-1}\beta\gamma\mu=\alpha^r$. Hence $\bar{\alpha}^r=\bar{\gamma}^{-1}\bar{\beta}\bar{\gamma}=\bar{\alpha}$ and, since α and $\bar{\alpha}$ have the same order, $\alpha=\alpha^r$. Therefore $\alpha=(\gamma\mu)^{-1}\beta(\gamma\mu)$ and $\alpha\sim\beta$.

The theorems.

THEOREM 1. *Let R be a finite local ring with maximal ideal M and $R/M=\mathrm{GF}(p^f)=k$. Let α, β be elements of $\mathrm{GL}_n(R)$ with $(|(\alpha)|,p)=1$ and $(|(\beta)|,p)=1$. Then α is similar to β if and only if α is similar to β modulo M.*

PROOF. This follows from the Observation by noting that the kernel, K, of $h:\mathrm{GL}_n(R)\to\mathrm{GL}_n(R/M)$ is solvable with cardinality a power of p.

THEOREM 2. *Let R be a finite commutative ring with identity and let the cardinality of R be m. Two elements α and β of $\mathrm{GL}_n(R)$ satisfying $(|(\alpha)|,m)=(|(\beta)|,m)=1$ are similar if and only if their canonical images over the Galois fields associated with R are similar.*

PROOF. This follows from Theorem 1 directly by means of the sequence of epimorphisms

$$\mathrm{GL}_n(R)=\prod_{i=1}^{t}\mathrm{GL}_n(R_i)\xrightarrow{\pi_i}\mathrm{GL}_n(R_i)\xrightarrow{h_i}\mathrm{GL}_n(k_i).$$

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