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A RENORMING OF NONREFLEXIVE BANACH SPACES

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ABSTRACT. Every nonreflexive Banach space can be equivalently renormed in such a way that it is not isometrically a conjugate space.

Dixmier [**1**] asked: "If X is isomorphic to a conjugate Banach space, is X isometric to a conjugate space?" Klee [**5**] gave a negative solution by giving an equivalent norm for l^∞ under which that space is not isometrically a dual space. Here, we show that such a norm exists for every nonreflexive Banach space. The result is precise since, obviously, if X is reflexive, it is isometrically a conjugate space under any equivalent renorming.

The prototype of our first lemma is the renorming theorem of Kadeć ([**2**], [**3**]) and Klee [**6**]. The version we give may have some novelty since we do not assume separability of X. We feel that the proof we sketch here is somewhat more revealing than existing proofs of the Kadeć-Klee theorem (cf. [**4**], [**7**] and [**8**, p. 486]).

LEMMA 1. *Suppose* $(X, \|\cdot\|)$ *is a Banach space*, Y *is a separable closed subspace of* X^*, *and that* Z *is a subspace of* X *which is norm determining over* Y (*that is*, $\sup\{y(z): z\in Z, \|z\|<1\}=\|y\|$ *for all* $y\in Y$). *Then, there is an equivalent norm*, $|||\cdot|||$, *on* X *such that, if* (x_δ^*) *is a net in* X^*, $y\in Y$, $x_\delta^*(z)\to y(z)$ *for each* $z\in Z$ *and* $|||x_\delta^*|||\to|||y|||$, *then* $\|x_\delta^*-y\|\to 0$.

SKETCH OF PROOF. Let $E_1\subset E_2\subset\cdots$ be a sequence of finite dimensional subspaces of Y with $\bigcup E_n$ dense in Y. For each x^* in X^*, define $|||x^*|||=\|x^*\|+\sum 2^{-n}\, d(x^*, E_n)$, where $d(x^*, E_n)$ is the usual distance from x^* to E_n. It is readily verified that each of the functions $x^*\to d(x^*, E_n)$ is weak* lower semicontinuous. It follows that the ball in $(X^*, |||\cdot|||)$ is weak* closed, so that $|||\cdot|||$ is the dual norm for some norm $|||\cdot|||$ on X, equivalent to $\|\cdot\|$.

Suppose that $(x_\delta^*)\subset X^*$, $y\in Y$ are as in the statement of the lemma. Since Z is norm determining over Y, one observes that $\liminf_\delta \|x_\delta^*\|\geqq\|y\|$ and, for each n,

$$\liminf_\delta d(x_\delta^*, E_n)\geqq d(y, E_n),$$

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so that in fact

$$\lim_{\delta} d(x_\delta^*, E_n) = d(y, E_n).$$

Since $d(y, E_n) \to 0$, one has $\|x_\delta^* - y\| \to 0$.

LEMMA 2. *If X is a nonreflexive Banach space, then X^* contains a separable subspace Y such that the natural map $T: X \to Y^*$ (defined by $(Tx)(y) = y(x)$ for all x, y) is not onto Y^*.*

PROOF. By Eberlein's theorem, the ball of X^* is not weakly countably compact, so there is a countable net (x_δ^*) in the ball of X^* which has no weakly convergent subnet. Let (x_λ^*) be a subnet which converges weak* to y in X^*. Let Y be the closed linear span of $(x_\delta^*) \cup (y)$. For each x in X, $(Tx)(x_\lambda^*) \to (Tx)(y)$, but $y^*(x_\lambda^*)$ fails to converge to $y^*(y)$ for some y^* in Y^* so $y^* \notin TX$.

THEOREM. *If $(X, \|\cdot\|)$ is a nonreflexive Banach space, there is an equivalent norm, $|||\cdot|||$, on X under which X fails to be isometric to a conjugate space.*

PROOF. Let Y be the subspace of X^* from Lemma 2, $Z = X$ and $|||\cdot|||$ the norm of Lemma 1. If $(X, |||\cdot|||)$ is isometric to W^*, then consider W as being canonically embedded in X^* $(= W^{**})$. Since the ball of W is weak* dense in that of W^{**}, for each y in Y, there is a net (w_δ) in W with $|||w_\delta||| \leqq |||y|||$ for all δ and $w_\delta(x) \to y(x)$ for each x in X. Since $\lim\inf |||w_\delta||| \geqq |||y|||$, $|||w_\delta||| \to |||y|||$ so, by Lemma 1, $w_\delta \to y$. Since W is a closed subspace, $Y \subset W$. However, the natural map of X to W^* must be onto which forces the natural map of X to Y^* to be onto. This contradicts the choice of Y, completing the proof.

We recall that X is isometric to a conjugate space if and only if there is a norm one projection P of X^{**} onto X such that $(I-P)X^{**}$ is weak* closed [1]. The weak* closedness of $(I-P)X^{**}$ is essential (viz., $L_1([0, 1])$). This raises the problem: *Can every nonreflexive space be renormed so that it is not the range of a norm one projection on its second conjugate space?*

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