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A NOTE ON NORMAL-OPERATOR-VALUED ANALYTIC FUNCTIONS[1]

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ABSTRACT. In this paper we prove that the set of values of a normal-operator-valued function, defined and analytic on an open connected set in the complex plane, is commutative.

The aim of this note is to prove that the set of values of a normal-operator-valued function, defined and analytic on an open connected set in the complex plane, is commutative. We were not able to find whether such a result is already known.

Throughout the paper we denote by $\mathcal{N}$ the set of nonnegative integers and by $L(X)$ the C^*-algebra of all bounded endomorphisms of a complex Hilbert space X.

LEMMA. *Let X be a complex Hilbert space and $\mathcal{D}$ the open unit disc in the complex plane. Let $F:\mathcal{D}\to L(X)$ be a normal-operator-valued analytic function given by*

$$(1) \qquad F(\zeta) = A_0 + A_1\zeta + A_2\zeta^2 + \cdots \qquad (\zeta \in \mathcal{D}).$$

Then

$$A_iA_j = A_jA_i \qquad (i \in \mathcal{N}, j \in \mathcal{N}),$$
$$A_iA_j^* = A_j^*A_i \qquad (i \in \mathcal{N}, j \in \mathcal{N}).$$

PROOF. By the normality assumption we have

$$(2) \qquad F(\zeta)F(\zeta)^* - F(\zeta)^*F(\zeta) = 0 \qquad (\zeta \in \mathcal{D}).$$

Since F is analytic on $\mathcal{D}$, the series

$$(3) \qquad \sum_{i=0}^{\infty} \|A_i\|\, r^i$$

converges for $0 \leq r < 1$ (cf. [2, Theorem 3.11.4]). This convergence and the isometricity of involution allow term-by-term multiplication in (2)

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which gives

$$\sum_{i=0}^{\infty}\left[\sum_{k=0}^{i}(A_kA_{i-k}^* - A_{i-k}^*A_k)\zeta^k\bar{\zeta}^{i-k}\right] = 0 \qquad (\zeta \in \mathscr{D}). \tag{4}$$

Since the series (3) converges for $0 \leqq r < 1$ it is easy to see that also the series

$$\sum_{i=0}^{\infty}\left[\sum_{k=0}^{i}\|A_kA_{i-k}^* - A_{i-k}^*A_k\|\right]r^i \tag{5}$$

converges for $0 \leqq r < 1$. It follows that the series (4) converges uniformly on every closed disc contained in $\mathscr{D}$.

Let $p \in \mathscr{N}$. Multiplying (4) with ζ^{p-1} and integrating term-by-term along the boundary of the disc with radius $r<1$ and with center at the point 0 we obtain

$$\sum_{i=0}^{\infty}(A_iA_{i+p}^* - A_{i+p}^*A_i)r^{2(i+p)} = 0 \qquad (0 \leqq r < 1). \tag{6}$$

The convergence of the series (5) implies that the series

$$G(\zeta) = \sum_{i=0}^{\infty}(A_iA_{i+p}^* - A_{i+p}^*A_i)\zeta^{2(i+p)}$$

is absolutely convergent on $\mathscr{D}$. This means that the function $\zeta \mapsto G(\zeta)$ is analytic on $\mathscr{D}$. Further, (6) tells us that $G(r)=0$ $(0 \leqq r < 1)$. By a well-known theorem it follows that $G(\zeta)=0$ $(\zeta \in \mathscr{D})$, which implies

$$A_iA_{i+p}^* - A_{i+p}^*A_i = 0 \qquad (i \in \mathscr{N}).$$

Since $p \in \mathscr{N}$ was arbitrary it follows that

$$A_iA_j^* = A_j^*A_i \qquad (i \in \mathscr{N}, j \in \mathscr{N}). \tag{7}$$

In particular, (7) tells that A_i $(i \in \mathscr{N})$ are normal operators. Now by the theorem of Fuglede (cf. [1, p. 934]), (7) implies $A_iA_j = A_jA_i$ $(i \in \mathscr{N}$, $j \in \mathscr{N})$, which, together with (7), proves the assertion. Q.E.D.

THEOREM. *Let X be a complex Hilbert space, $\mathscr{D}$ an open connected set in the complex plane and let $F:\mathscr{D} \to L(X)$ be an analytic function. Let a neighborhood $\mathscr{U}(\zeta_0)$ of a point $\zeta_0 \in \mathscr{D}$ exist such that $F(\zeta)$ $(\zeta \in \mathscr{U}(\zeta_0))$ are normal operators. Then $F(\zeta)$ is a normal operator for every $\zeta \in \mathscr{D}$ and*

$$F(\zeta_1)F(\zeta_2) = F(\zeta_2)F(\zeta_1) \qquad (\zeta_1 \in \mathscr{D}, \zeta_2 \in \mathscr{D}).$$

PROOF. With no loss of generality we may assume that $\mathscr{U}(\zeta_0)$ is an open disc with radius r and with center at the point ζ_0. Expand F into the

Taylor series

$$F(\zeta) = A_0 + A_1(\zeta - \zeta_0) + A_2(\zeta - \zeta_0)^2 + \cdots.$$

Clearly we may assume that $\zeta_0=0$ and $r=1$, and by the Lemma we get

$$A_iA_j = A_jA_i \qquad (i \in \mathscr{N}, j \in \mathscr{N}),$$
$$A_iA_j^* = A_j^*A_i \qquad (i \in \mathscr{N}, j \in \mathscr{N}).$$

This assures (cf. [**3**, p. 182]) the existence of a commutative C^*-subalgebra $\mathscr{A}$ of the algebra $L(X)$ which contains the operators A_i $(i \in \mathscr{N})$. Since $\mathscr{A}$ is a closed linear subspace of $L(X)$, by the Hahn-Banach theorem a set $\mathscr{T}$ of bounded linear functionals on $L(X)$ exists such that $A \in \mathscr{A}$ if and only if $u(A)=0$ $(u \in \mathscr{T})$. Now, clearly $F(\zeta) \in \mathscr{A}$ $(\zeta \in \mathscr{U}(\zeta_0))$ so that $u[F(\zeta)]=0$ $(\zeta \in \mathscr{U}(\zeta_0),\ u \in \mathscr{T})$. By the analytic continuation principle the connectedness of $\mathscr{D}$ implies that $u[F(\zeta)]=0$ $(\zeta \in \mathscr{D},\ u \in \mathscr{T})$ which means that $F(\zeta) \in \mathscr{A}$ $(\zeta \in \mathscr{D})$. Since $\mathscr{A}$ is a commutative C^*-subalgebra of $L(X)$, the assertion is proved. Q.E.D.

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