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REPRESENTATION THEOREMS FOR COMPACT OPERATORS

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ABSTRACT. It is shown that c_0 (the Banach space of zero-convergent sequences) is the only Banach space with basis that satisfies the following property: For every compact operator $T:c_0 \to E$ from c_0 into a Banach space E, there is a sequence λ in c_0 and an unconditionally summable sequence $\{y_n\}$ in E such that $T\mu = \sum \lambda_n \mu_n y_n$ for each μ in c_0. This result is then used to show that a linear operator $T:E \to F$ from a locally convex space E into a Fréchet space F has a representation of the form $Tx = \sum \lambda_n \langle x, a_n \rangle y_n$, where λ is a sequence in c_0, $\{a_n\}$ is an equicontinuous sequence in the topological dual E' of E and $\{y_n\}$ is an unconditionally summable sequence in F, if and only if T can be "compactly factored" through c_0.

A linear operator $T:E \to F$ from one locally convex space into another is precompact [respectively, compact] if T transforms a neighborhood of 0 in E into a precompact [respectively, relatively compact] subset of F. (All of our spaces are assumed to be Hausdorff.)

A sequence $\{y_n\}$ in a sequentially complete locally convex space E is unconditionally summable if $\sum \xi_n y_n$ converges for each ξ in l_∞. (l_∞ is the Banach space of bounded sequences equipped with the supremum norm $\|\cdot\|_\infty$.) For conditions equivalent to unconditional summability we refer the reader to [3, Chapter IV, §1, pp. 58–59]. A sequence $\{y_n\}$ in a locally convex space E is weakly unconditionally summable if $\sum |\langle y_n, b \rangle| < \infty$ for each b in the topological dual E' of E.

Let q be a seminorm on a locally convex space E. Let $E(q) = E/q^{-1}(0)$, and let $K_q:E \to E(q)$ denote the natural map. Equip $E(q)$ with the norm $\|K_q(x)\| = q(x)$. Following [6, 1.2] we say that q is precompact if the natural map K_q is precompact. In [6, 2.5] we showed that q is a precompact seminorm on E if and only if there is a sequence λ in c_0 and an equicontinuous sequence $\{a_n\}$ in E' such that, for each x in E,

$$q(x) \leqq \sup |\lambda_n| \, |\langle x, a_n \rangle|.$$

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In [**6**, 2.10, p. 92] we showed that a linear operator $T: E \to F$ from a locally convex space E into a normed space F is precompact if and only if there is a precompact seminorm q on E such that $\|Tx\| \leqq q(x)$ for each x in E. (In the normed space case this result has also been given by Terzioglu [**9**, (1), p. 93] and [**8**, Theorem 1, p. 76].)

The following lemma is an immediate consequence of [**2**, Theorem 2.4, p. 477].

1. LEMMA. *A seminorm q on c_0 is precompact if and only if there is a sequence λ in c_0 such that, for each μ in c_0, $q(\mu) \leqq \sup|\lambda_n|\,|\mu_n|$.*

2. PROPOSITION. *A linear operator $T: c_0 \to E$ from c_0 into a normed space E is precompact if and only if there is a sequence λ in c_0 and a sequence $\{y_n\}$ in E, that is (weakly) unconditionally summable in the completion of E, such that, for each μ in c_0, $T\mu = \sum \lambda_n \mu_n y_n$.*

PROOF. Lemma 1 and [**1**, Lemma 2, p. 159] show that the condition is sufficient. We now prove necessity.

Suppose $T: c_0 \to E$ is precompact; then (see Lemma 1) there is a sequence λ in c_0 such that, for each μ in c_0,

$$\|T\mu\| \leqq \sup |\lambda_n|^2\,|\mu_n|.$$

We assume that each $\lambda_n \neq 0$. For each n, let $y_n = T[(\lambda_n)^{-1} e_n]$ (where $\{e_n\}$ is the usual unit vector basis of c_0). Since

$$\left\| \sum_{k=n}^{m} \xi_k y_k \right\| = \left\| T\left[\sum_{k=n}^{m} \xi_k (\lambda_k)^{-1} e_k \right] \right\| \leqq \sup\{|\lambda_k|\,|\xi_k| : k \geqq n\}$$

for each ξ in l_∞, it follows that $\{y_n\}$ is unconditionally summable in the completion of E. Since

$$\left\| T\mu - \sum_{k=1}^{n} \lambda_k \mu_k y_k \right\| = \left\| T\left[\sum_{k>n} \mu_k e_k \right] \right\| \leqq \sup\{|\lambda_k|^2 |\mu_k| : k > n\}$$

for each μ in c_0, it follows that $T\mu = \sum \lambda_n \mu_n y_n$.

The following theorem is closely related to Theorem 2.18 in [**6**] and shows that (in the Fréchet space case) the Schwartz maps introduced in [**6**, 2.15] coincide with the linear operators that can be "compactly factored" through c_0. Recall that a Fréchet space is a complete metrizable locally convex space.

3. THEOREM. *Let $T: E \to F$ be a linear operator from a locally convex space E into a Fréchet space F. The following are equivalent:*

(a) *T has a representation of the form*

$$Tx = \sum \lambda_n \langle x, a_n \rangle y_n$$

where λ is a sequence in c_0, $\{a_n\}$ is an equicontinuous sequence in E' and $\{y_n\}$ is an unconditionally summable sequence in F.

(b) *Same as* (a) *except that $\{y_n\}$ is weakly unconditionally summable.*

(c) *There exist compact linear operators $P: E \to c_0$ and $Q: c_0 \to F$ such that $T = QP$.*

(d) *Same as* (c) *except that Q is not assumed to be compact, but only continuous.*

(e) *Same as* (c) *except that P is not assumed to be compact, but only continuous.*

PROOF. (a) implies (b) is clear.

(b) implies (d). Suppose that $Tx = \sum \lambda_n \langle x, a_n \rangle y_n$ where λ is a sequence in c_0, $\{a_n\}$ is an equicontinuous sequence in E' and $\{y_n\}$ is a weakly unconditionally summable sequence in F. By [**5**, Satz 1.3.5, p. 24] it follows that $\sum \mu_n y_n$ converges in F, whenever μ is a sequence in c_0. Let $P: E \to c_0$ be defined by $P(x) = \{\lambda_n \langle x, a_n \rangle\}$; and let $Q: c_0 \to F$ be defined by $Q(\mu) = \sum \mu_n y_n$. Then $T = QP$ and by using [**4**, III.2,3, p. 85] it is easy to show that P is compact. (Since $\{a_n\}$ is equicontinuous, P is continuous.)

(d) implies (c) follows easily from [**6**, Theorem 2.18] and the fact that c_0 has the c_0-extension property (see [**6**, 2.17]).

(c) implies (e) is clear.

(e) implies (a) follows from Proposition 2, [**6**, 2.11] and [**6**, 2.19].

4. REMARK. Compact linear operators having representations of the form described in Theorem 3(a) have been studied by Terzioglu [**8**] and [**9**], and independently by the author [**6**]. By using the results of [**6**] it is easy to see that Theorem 3 remains valid if the hypothesis, that F be a Fréchet space, is replaced by the weaker hypothesis that F be a sequentially complete locally convex space satisfying any one of the conditions of [**6**, 2.12]. By modifying the proof of Proposition 2 we can show that (given $\varepsilon > 0$) every compact linear operator $T: c_0 \to E$ from c_0 into a Banach space E has a representation of the form described in Theorem 3 with $\|\lambda\|_\infty \leqq \|T\| + \varepsilon$ and $\|\sum \xi_n y_n\| \leqq \|\xi\|_\infty$ for all ξ in l_∞.

We now show that the condition in Proposition 2 actually characterizes the Banach space c_0. Recall that $\{x_n\}$ is a normalized basis for a Banach space E if $\|x_n\| = 1$ and each x in E has a unique representation of the form $x = \sum \alpha_n x_n$. The sequence $\{f_n\}$ of linear forms defined by $x = \sum \langle x, f_n \rangle x_n$ is called the associated sequence of coefficient forms. It is well known (see [7, Theorem 3.1, p. 20]) that $\sup \|f_n\| < \infty$.

Let $\{x_n\}$ and $\{y_n\}$ be normalized bases for Banach spaces E and F, respectively. The bases $\{x_n\}$ and $\{y_n\}$ are equivalent if for each sequence ξ in l_∞ the series $\sum \xi_n x_n$ converges if and only if the series $\sum \xi_n y_n$ converges. If $\{x_n\}$ is equivalent to $\{y_n\}$, then (see [**1**, 1.2, p. 152]) E and F are isomorphic Banach spaces. In fact, (see [7, Theorem 8.1(d), pp. 69–70]) the function $T(x_n)=y_n$ can be extended to a continuous linear operator from E onto F.

5. THEOREM. *Let E be a Banach space with normalized basis $\{x_n\}$. Let $\{f_n\}$ denote the associated sequence of coefficient forms. The following are equivalent*:

(a) *$\{x_n\}$ is equivalent to the unit vector basis of c_0.*

(b) *For each compact linear operator $T:E\to F$ from E into a Banach space F, there is a sequence λ in c_0 and an unconditionally summable sequence $\{y_n\}$ in F such that, for each x in E, $Tx=\sum \lambda_n\langle x,f_n\rangle y_n$.*

(c) *Same as* (b) *except that $\{y_n\}$ is weakly unconditionally summable.*

(d) *For each compact linear operator $T:E\to F$ from E into a Banach space F, there is a sequence λ in c_0 such that, for each x in E,*

$$\|Tx\| \leqq \sup |\lambda_n|\,|\langle x,f_n\rangle|.$$

(e) *For each precompact seminorm q on E there is a sequence λ in c_0 such that, for each x in E,*

$$q(x) \leqq \sup |\lambda_n|\,|\langle x,f_n\rangle|.$$

PROOF. (a) implies (b) follows from Proposition 3. (b) implies (c) is trivial. (c) implies (d) is an easy consequence of [**1**, Lemma 2, p. 159]. (d) implies (e) follows from the fact that $q(x)=\|K_q(x)\|$ (see the introduction).

(e) implies (a). By [**1**, Lemma 3, p. 160] it suffices to show that $\sum |\langle x_n, a\rangle|<\infty$ for each a in E'. Let a be an element of E'; then (by (e)) there is a sequence λ in c_0 such that for each x in E

$$(*)\qquad |\langle x, a\rangle| \leqq \sup |\lambda_n|\,|\langle x,f_n\rangle|.$$

For each n let α_n be a scalar such that $\alpha_n\langle x_n, a\rangle=|\langle x_n, a\rangle|$ and $|\alpha_n|=1$. Let $x=\alpha_1x_1+\cdots+\alpha_kx_k$; then (by (*))

$$|\langle x_1, a\rangle| + \cdots + |\langle x_k, a\rangle| = |\langle x, a\rangle| \leqq \sup |\lambda_n|\, M,$$

where $M=\sup\|f_n\|$. Therefore, $\sum |\langle x_n, a\rangle|<\infty$ for a in E' and $\{x_n\}$ is equivalent to the unit vector basis of c_0.

ADDED IN PROOF. It has recently been brought to our attention that in the Banach space case Grothendieck (using his theory of tensor products) has essentially proven Proposition 2. See *Sur certaines classes de suites dans les espaces de Banach, et le théorème de Dvoretzky-Rogers*, Bol. Soc. Mat. São Paulo **8** (1953), 81–110 (page 88).

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