PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 2, February 1973

CONVEX METRIC SPACES WITH 0-DIMENSIONAL MIDSETS

L. D. LOVELAND AND J. E. VALENTINE

ABSTRACT. Let X be a nontrivial, complete, convex, locally externally convex metric space. Assuming that the midset of each pair of points of X is 0-dimensional and that any nonmaximal metric segment that intersects a midset twice lies in that midset, we show that X is isometric to either the euclidean line E^1 or to a 1-dimensional spherical space $S_{1,\alpha}$ (the circle of radius α in the euclidean plane with the "shorter arc" metric).

The midset of two distinct points a and b in a metric space is defined as the set of all points x in the space for which the distances ax and bx are equal. A metric space X is said to have the weak linear midset property (WLMP) if, for each pair of its distinct points a and b, a nonmaximal (with respect to inclusion) metric segment S belongs to the midset $M(a, b)$ whenever $S\cap M(a, b)$ contains two points. If, in addition to the WLMP, each midset of a space X is a 0-dimensional set, then we say that X has the 0-dimensional weak linear midset property (0-WLMP). We use the 0-WLMP to characterize the euclidean line E^1 and 1-dimensional spherical space $S_{1,\alpha}$ among complete, convex, and locally externally convex metric spaces. A 1-dimensional spherical space $S_{1,\alpha}$ is the circle of radius α in the euclidean plane under the "shorter arc" metric.

Berard [2] characterized a topological simple closed curve among convex complete metric spaces with the condition that each midset consist of two points—the double midset property (DMP). We show that Berard's conditions actually yield a characterization of a 1-dimensional spherical space.

In another paper, Berard [1] showed that, among connected metric spaces, intervals are the only spaces in which all midsets are singleton sets. This condition on the midsets was called the unique midset property and is abbreviated UMP. Busemann [4] has given characterizations of euclidean, hyperbolic, and spherical spaces among his G-spaces by imposing convexity on the midsets.

Received by the editors May 30, 1972.

AMS (MOS) subject classifications (1969). Primary 5250, 5435.

Key words and phrases. Convex, locally externally convex, midsets, weak linear midset property, 0-dimensional midsets.

A metric space X is called locally externally convex if, for each of its points p, there is a neighborhood N of p such that if x and y belong to N, then there exists a point z in N such that xyz (xyz means that $x \neq y \neq z$ and $xy + yz = xz$). For other definitions and notation the reader is referred to [3] and [4].

THEOREM 1. *If X is a nontrivial, convex, locally externally convex, complete, metric space with the UMP, then X is isometric to the euclidean line E^1.*

PROOF. The convexity of X implies that X is connected, so the conditions given by Berard in [1] are satisfied. Thus, by [1], X is homeomorphic either to a closed interval, a closed ray, or to E^1. Of course the local external convexity on X rules out both the interval and the ray. Now it is an easy exercise to prove that a convex, complete, metric space is isometric to E^1 if it is homeomorphic to E^1.

The following theorem is an extension of the characterization given in [2] by Berard.

THEOREM 2. *If X is a nontrivial, convex, complete, metric space with the DMP, then X is isometric to a* 1*-dimensional spherical space.*

PROOF. It follows from [2] that X is homeomorphic to a circle, so it remains to produce an isometry between X and a round circle. If a and x are two points of X, it follows from the fact that X is convex and compact that there is a maximal segment $S(a, b)$ in X containing x. Now let $\{x_n\}$ be a sequence of points in $X - S(a, b)$ converging to b, and let N be an integer such that for $n > N$ we have ax_1x_n. From the convexity of X and the continuity of the metric, it follows that ax_1b. Thus the union of the two segments $S(a, x_1)$ and $S(x_1, b)$ is a segment $S'(a, b)$ (see Lemma 15.1 of [3]). Since $S(a, b) \cap S'(a, b) = \{a, b\}$ and X is a simple closed curve, we see that $X = S'(a, b) \cup S(a, b)$. Let C be the circle in E^2 given by the equation $x^2 + y^2 = (ab/\pi)^2$ with the "shorter arc" metric. It is clear that if H and H' are two semicircles whose union is C, then isometries exist between $S(a, b)$ and H and between $S'(a, b)$ and H'. Let f denote the injective map of X onto C defined by these isometries, and let $f(x)$ be denoted by x'. It will be clear that f is an isometry once we show that f preserves distances for two points x and y in the interiors of $S(a, b)$ and $S'(a, b)$, respectively. We may assume that xay. If $x'a'y'$ holds we have $xy = x'y'$. It can be shown that $x'b'y'$ implies xby. Thus $xy = x'y'$.

THEOREM 3. *If X is a nontrivial, complete, convex, locally externally convex, metric space with the* 0*-WLMP then X is isometric either to the euclidean line E^1 or to a* 1*-dimensional spherical space.*

PROOF. We shall show that either X has the DMP or the UMP, so that Theorem 3 will follow from Theorems 1 and 2. Our first step is to show that X contains no ramification points. A point q is called a ramification point if there exist distinct points p, r, r' such that q is a midpoint of p and r and q is also a midpoint of p and r'. In the setting of this theorem a ramification point of X would lie in each of the distinct segments $S(p, r)$ and $S(p, r')$, and no generality is lost in assuming that $qr=qr'$. But this puts p and q in $M(r, r')\cap S(p, r)$, and it follows from the WLMP that the segment $S(p, r)$ lies in $M(r, r')$. This contradicts the fact that $M(r, r')$ is 0-dimensional, and we see that X has no ramification points.

Suppose now that there exist two distinct points a and b in X such that $M(a, b)$ consists of the point m. Since there are no ramification points in X it follows that every point t of X must be linear with a and b. Thus X has the UMP and, from Theorem 1, we see that X is isometric with E^1.

If no pair of distinct points of X has a singleton set as a midset, then every midset contains at least two points. We now show that under this hypothesis, X has the DMP. Suppose to the contrary that some distinct pair of points a and b has a midset containing at least three points x, y, and z. From the 0-WLMP it follows that each segment having endpoints in $M(a, b)$ is maximal and intersects $M(a, b)$ at no other points. Let S_1, S_2, and S_3 be three distinct maximal segments with endpoints exhausting each pair in the set $\{x, y, z\}$. We may assume a labeling so that two of these segments, say S_1 and S_2, lie, except for their endpoints, closer to a than to b, and we may assume that $S_1\cap S_2=\{z\}$, since there are no ramifications in X. Let $\{x_i\}$ and $\{y_i\}$ be sequences in S_1 and S_2, respectively, converging to z. Now, for each i, the segments $S(b, x_i)$ and $S(b, y_i)$ must intersect $M(a, b)$ exactly once. A positive integer N exists such that, for $i>N$, neither $S(b, x_i)$ nor $S(b, y_i)$ contains x or y, for otherwise the continuity of the metric would yield bxz or byz which, in light of Lemma 15.1 of [3], contradicts the fact that both S_1 and S_2 are maximal. Furthermore not both of these segments, for any i, can contain z, for then z would be a ramification point of X. Thus for each $i>N$ there is a point m_i in $M(a, b)$ and in either $S(b, x_i)$ or $S(b, y_i)$ such that m_i is not in the set $\{x, y, z\}$. Similar reasoning makes it clear that $\{m_i\}$ contains infinitely many distinct points. Since both $\{x_i\}$ and $\{y_i\}$ converge to z, the continuity of the metric insures that $\{m_i\}$ converges to z. Now, for each i, there is a maximal segment $S(z, m_i)$. Of course the sequence $\{zm_i\}$ of lengths of these segments converges to zero, and this contradicts the local external convexity of X. Thus in this case X has the DMP, and it follows from Theorem 2 that X is isometric to a 1-dimensional spherical space.

REFERENCES

1. Anthony D. Berard, Jr., *Characterizations of metric spaces by the use of their midsets: Intervals*, Fund. Math. **73** (1971), 1–7.

2. ———, *Characterizations of metric spaces by the use of their midsets: One-spheres*, Fund. Math. (to appear).

3. L. M. Blumenthal, *Theory and applications of distance geometry*, Clarendon Press, Oxford, 1953. MR **14**, 1009.

4. H. Busemann, *The geometry of geodesics*, Academic Press, New York, 1955. MR **17**, 779.

DEPARTMENT OF MATHEMATICS, UTAH STATE UNIVERSITY, LOGAN, UTAH 84321