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A NEW PROOF OF THE FUNCTIONAL EQUATION OF DIRICHLET L-FUNCTIONS[1]

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ABSTRACT. A simple proof, using contour integration, of the functional equation of Dirichlet L-functions is given.

Let χ be a nonprincipal, primitive character modulo k. Let

$$G(z,\chi)=\sum_{j=1}^{k-1}\chi(j)e^{2\pi ijz/k}$$

denote a Gaussian sum, and put $G(\chi)=G(1,\chi)$. We shall need two fundamental properties of Gaussian sums. If n is an integer [1, p. 312],

$$G(n,\bar{\chi})=\chi(n)G(\bar{\chi}). \tag{1}$$

Secondly [1, p. 313],

$$G(\chi)G(\bar{\chi})=\chi(-1)k. \tag{2}$$

THEOREM. *The Dirichlet L-function,*

$$L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s},\qquad \sigma=\operatorname{Re}s>0,$$

can be analytically continued to an entire function which satisfies the functional equation

$$L(1-s,\chi)=(k/2\pi)^{s}k^{-1}G(\chi)\Gamma(s)L(s,\bar{\chi})\{e^{-\pi is/2}+\chi(-1)e^{\pi is/2}\}. \tag{3}$$

PROOF. For $\sigma>1$, it is quite easy to show that [2, pp. 194, 200]

$$\Gamma(s)L(s,\chi)=\int_0^{\infty}\frac{x^{s-1}G(ikx/2\pi,\chi)\,dx}{1-e^{-kx}}. \tag{4}$$

Equation (4) is the starting point for a proof of (3) by Ayoub [2], but otherwise our proof has nothing in common with his.

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Assume that s is real and $s>1$. If m is a positive integer, let C_m denote the positively oriented, closed contour consisting of Γ_m, the right half of the circle with center $(0, 0)$ and radius $m+\frac{1}{2}$, together with the vertical diameter indented at the origin by a semicircle Γ_ε of radius $\varepsilon<1$ in the right half plane. Define

$$F(z) = \pi e^{-\pi i z}G(z, \bar{\chi})/G(\bar{\chi})z^s \sin(\pi z),$$

where z^s is given its principal value. On the interior of C_m, F is analytic except for simple poles at $z=1, \cdots, m$. The residue of F at the positive integer n is $G(n, \bar{\chi})/G(\bar{\chi})n^s=\chi(n)n^{-s}$, upon the use of (1). Hence, by the residue theorem,

$$\frac{1}{2\pi i}\int_{C_m} F(z)\,dz = \sum_{n=1}^{m} \chi(n)n^{-s}. \tag{5}$$

Now, $|e^{-\pi i z}G(z, \bar{\chi})/\sin(\pi z)|$ has period k and tends to zero exponentially as $\operatorname{Im} z$ tends to $\pm\infty$. Thus, there exists a positive number M, independent of m, such that for all z on Γ_m,

$$|e^{-\pi i z}G(z, \bar{\chi})/\sin(\pi z)| \leqq M.$$

Since $s>1$, clearly the integral of F over Γ_m tends to 0 as m tends to ∞. Hence, upon letting m tend to ∞ in (5), we find that

$$\begin{aligned} L(s, \chi) = &\int_{i\varepsilon}^{i\infty} \frac{G(z, \bar{\chi})\,dz}{G(\bar{\chi})z^s(1 - e^{2\pi i z})} + \int_{-i\varepsilon}^{-i\infty} \frac{e^{-2\pi i z}G(z, \bar{\chi})\,dz}{G(\bar{\chi})z^s(1 - e^{-2\pi i z})} \\ &+ \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} F(z)\,dz. \end{aligned} \tag{6}$$

The two infinite integrals on the right side of (6) each converge uniformly on any compact set of the complex s-plane. Thus, (6) shows that $L(s, \chi)$ can be analytically continued to an entire function of s, and (6) is then valid for all s.

Now suppose that $s<0$. Since $G(0, \chi)=0$, it is trivial to see that the integral over Γ_ε on the right side of (6) tends to 0 as ε tends to 0. Letting ε tend to 0 in (6), we then obtain for $s<0$,

$$\begin{aligned} L(s, \chi) &= ie^{-\pi i s/2}\int_0^\infty \frac{G(iy, \bar{\chi})\,dy}{G(\bar{\chi})y^s(1 - e^{-2\pi y})} - ie^{\pi i s/2}\int_0^\infty \frac{e^{-2\pi y}G(-iy, \bar{\chi})\,dy}{G(\bar{\chi})y^s(1 - e^{-2\pi y})} \\ &= ie^{-\pi i s/2}(k/2\pi)^{1-s}\int_0^\infty \frac{G(iky/2\pi, \bar{\chi})\,dy}{G(\bar{\chi})y^s(1 - e^{-ky})} \\ &\quad - ie^{\pi i s/2}(k/2\pi)^{1-s}\int_0^\infty \frac{e^{-ky}G(-iky/2\pi, \bar{\chi})\,dy}{G(\bar{\chi})y^s(1 - e^{-ky})}. \end{aligned} \tag{7}$$

If in the definition of $G(z,\chi)$ we replace j by $k-j$, we find that

$$e^{-ky}G(-iky/2\pi, \bar{\chi}) = \chi(-1)G(iky/2\pi, \bar{\chi}).$$

Hence, with the use of (4), (7) reduces to

$$L(s, \chi) = i(k/2\pi)^{1-s}\Gamma(1 - s)L(1 - s, \bar{\chi})\{e^{-\pi is/2} - \chi(-1)e^{\pi is/2}\}/G(\bar{\chi}). \tag{8}$$

If we replace s by $1-s$, apply (2), and use analytic continuation, (8) reduces to (3), and the proof is complete.

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