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DISTRIBUTION OF SEMI-k-FREE INTEGERS

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ABSTRACT. Let $Q_k^*(x)$ denote the number of semi-k-free integers $\leqq x$. It is known that $Q_k^*(x)=\alpha_k^* x+O(x^{1/k})$, where α^*_k is a constant. In this paper we prove that

$$\Delta_k^*(x) = Q_k^*(x) - \alpha_k^* x = O(x^{1/k} \exp\{-A \log^{3/5} x(\log\log x)^{-1/5}\}),$$

where A is an absolute positive constant. Further, on the assumption of the Riemann hypothesis, we prove that

$$\Delta_k^*(x) = O(x^{2/(2k+1)} \exp\{A \log x(\log\log x)^{-1}\}).$$

1. **Introduction.** Let k be a fixed integer $\geqq 2$. A positive integer n is called semi-k-free, if the multiplicity of each prime factor of n is not equal to k or, equivalently, if n is not divisible *unitarily* by the kth power of any prime. By a unitary divisor we mean, as usual, a divisor $d>0$ of n such that $(d, n/d)=1$. The integer 1 is also considered to be semi-k-free. Let Q_k^* denote the set of semi-k-free integers and let $q_k^*(n)$ denote the characteristic function of the set of semi-k-free integers, that is, $q_k^*(n)=1$ or 0 according as $n \in Q_k^*$ or $n \notin Q_k^*$. Let x denote a real variable $\geqq 1$ and let $Q_k^*(x)$ denote the number of semi-k-free integers $\leqq x$. Recently, the first author [3] proved that

$$Q_k^*(x) = \alpha_k^* x + O(x^{1/k}), \tag{1.1}$$

where

$$\alpha_k^* = \prod_p \left(1 - \frac{1}{p^k} + \frac{1}{p^{k+1}}\right), \tag{1.2}$$

the product being extended over all primes p.

The object of the present paper is to improve the order of the error term in (1.1) to $O(x^{1/k}\delta(x))$, where $\delta(x)=\exp\{-A \log^{3/5}x(\log\log x)^{-1/5}\}$, A being a positive constant. We further improve the above order estimate to $O(x^{2/(2k+1)}\omega(x))$, on the assumption of the Riemann hypothesis, where $\omega(x)=\exp\{A \log x(\log\log x)^{-1}\}$, A being a positive constant.

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2. **Preliminaries.** Let $\mu(n)$ and $\varphi(n)$ respectively denote the Möbius function and the Euler totient function. Let $\sigma_s^*(n)$ denote the sum of the sth powers of the square-free divisors of n. Let $\varphi(x, n)$ denote the Legendre totient function, which is defined to be the number of positive integers $\leqq x$ which are relatively prime to n. It is known (cf. [**1**, Lemma 3.4]) that for $x \geqq 1$,

$$\varphi(x, n) = x\varphi(n)/n + O(\tau(n)), \tag{2.1}$$

where the O-constant is independent of n and x, $\tau(n)$ being the number of divisors of n.

REMARK. Hereafter, all the constants implied in the O-symbols are independent of n and x.

It is also known (cf. [**3**, Lemma 2]) that

$$\alpha_k^* \equiv \sum_{n=1}^{\infty} \frac{\mu(n)\varphi(n)}{n^{k+1}} = \prod_p \left(1 - \frac{1}{p^k} + \frac{1}{p^{k+1}}\right). \tag{2.2}$$

It can be easily shown by using standard arguments that

$$\sum_{n \leqq x} \frac{\sigma_{-s}^*(n)}{n^u} = O(x^{1-u}), \quad \text{for } s > 0 \text{ and } 0 \leqq u < 1. \tag{2.3}$$

We need the following

LEMMA 2.1. (Cf. [**5**, Lemma 3.5].) *For $x \geqq 3$ and for every $\varepsilon > 0$,*

$$M_n(x) \equiv \sum_{m \leqq x; (m,n)=1} \mu(m) = O(\sigma_{-1+\varepsilon}^*(n)x\delta(x)), \tag{2.4}$$

where

$$\delta(x) = \exp\{-A \log^{3/5} x(\log\log x)^{-1/5}\}, \tag{2.5}$$

A being a positive constant.

LEMMA 2.2. (Cf. [**5**, Lemma 5.2].) *If the Riemann hypothesis is true, then, for $x \geqq 3$ and for every $\varepsilon > 0$,*

$$M_n(x) \equiv \sum_{m \leqq x; (m,n)=1} \mu(m) = O(\sigma_{-1/2+\varepsilon}^*(n)x^{1/2}\omega(x)), \tag{2.6}$$

where

$$\omega(x) = \exp\{A \log x(\log\log x)^{-1}\}, \tag{2.7}$$

A being a positive constant.

LEMMA 2.3. *For $x \geqq 3$ and for every $\varepsilon > 0$,*

$$\sum_{m \leqq x; (m,n)=1} \mu(m)m = O(\sigma_{-1+\varepsilon}^*(n)x^2\delta(x)). \tag{2.8}$$

If the Riemann hypothesis is true, then

$$\sum_{m\leqq x;(m,n)=1} \mu(m)m = O(\sigma^*_{-1/2+\varepsilon}(n)x^{3/2}\omega(x)). \tag{2.9}$$

PROOF. (2.8) and (2.9) follow by partial summation and making use of (2.4) and (2.6) respectively.

LEMMA 2.4. *For* $x\geqq 3$,

$$N(x) \equiv \sum_{n\leqq x} \mu(n)\varphi(n) = O(x^2\delta(x)). \tag{2.10}$$

If the Riemann hypothesis is true, then

$$N(x) \equiv \sum_{n\leqq x} \mu(n)\varphi(n) = O(x^{3/2}\omega(x)). \tag{2.11}$$

PROOF. Since $\varphi(n)=\sum_{d\delta=n}\mu(d)\delta$ (cf. [2, (16.3.1)]), we have, by (2.8),

$$\begin{aligned}\sum_{n\leqq x}\mu(n)\varphi(n) &= \sum_{d\delta\leqq x}\mu(d\delta)\mu(d)\delta = \sum_{d\delta\leqq x;(d,\delta)=1}\mu(d)\mu(\delta)\mu(d)\delta \\ &= \sum_{d\leqq x}\mu^2(d)\sum_{\delta\leqq x/d;(\delta,d)=1}\mu(\delta)\delta \\ &= O\left(\sum_{d\leqq x}\mu^2(d)\sigma^*_{-1+\varepsilon}(d)\frac{x^2}{d^2}\,\delta\left(\frac{x}{d}\right)\right) \\ &= O\left(x^2\sum_{n\leqq x}\frac{\sigma^*_{-1+\varepsilon}(n)}{n^2}\,\delta\left(\frac{x}{n}\right)\right).\end{aligned}$$

We note that $x^\varepsilon\delta(x)$ is monotonic increasing for every $\varepsilon>0$, so that we have

$$\begin{aligned}\sum_{n\leqq x}\frac{\sigma^*_{-1+\varepsilon}(n)}{n^2}\,\delta\left(\frac{x}{n}\right) &= \frac{1}{x^\varepsilon}\sum_{n\leqq x}\frac{\sigma^*_{-1+\varepsilon}(n)}{n^{2-\varepsilon}}\left(\frac{x}{n}\right)^\varepsilon\delta\left(\frac{x}{n}\right) \\ &\leqq \frac{1}{x^\varepsilon}\cdot x^\varepsilon\delta(x)\sum_{n\leqq x}\frac{\sigma^*_{-1+\varepsilon}(n)}{n^{2-\varepsilon}} \\ &\leqq \delta(x)\sum_{n\leqq x}\frac{\tau(n)}{n^{2-\varepsilon}} = O(\delta(x)),\end{aligned}$$

since $\tau(n)=O(n^\varepsilon)$ for every $\varepsilon>0$ (cf. [2, Theorem 315]). Hence

$$\sum_{n\leqq x}\mu(n)\varphi(n) = O(x^2\delta(x)),$$

so that (2.10) follows.

Making use of the same argument adopted above, but using (2.9) instead of (2.8), we get (2.11).

LEMMA 2.5. *For* $x \geqq 3$, $s>2$,

$$\sum_{n>x} \frac{\mu(n)\varphi(n)}{n^s} = O\left(\frac{\delta(x)}{x^{s-2}}\right). \tag{2.12}$$

If the Riemann hypothesis is true, then

$$\sum_{n>x} \frac{\mu(n)\varphi(n)}{n^s} = O\left(\frac{\omega(x)}{x^{s-3/2}}\right). \tag{2.13}$$

PROOF. Putting $f(n)=1/n^s$, it is easy to show that $f(n+1)-f(n)=O(1/n^{s+1})$. Therefore by partial summation we have

$$\begin{aligned}\sum_{n>x} \frac{\mu(n)\varphi(n)}{n^s} &= \sum_{n>x} \mu(n)\varphi(n)f(n) \\ &= -N(x)f([x]+1) - \sum_{n>x} N(n)\{f(n+1)-f(n)\}.\end{aligned} \tag{2.14}$$

Substituting in (2.14), the O-estimate of $N(x)$ obtained in (2.10), we get (2.12) after some simple manipulation as in (cf. [**4**, Lemma 2.2]).

Similarly, substituting in (2.14), the O-estimate of $N(x)$ obtained in (2.11), we get (2.13).

3. **Main results.** In this section we prove the following.

THEOREM 3.1. *For* $x \geqq 3$,

$$Q_k^*(x) = \alpha_k^* x + O(x^{1/k}\delta(x)), \tag{3.1}$$

where α_k^* *is given by* (1.2) *and* $\delta(x)$ *is given by* (2.5).

PROOF. We have (cf. [**3**]) that $q_k^*(n) = \sum_{d^k\delta=n;(d,\delta)=1} \mu(d)$. Hence

$$Q_k^*(x) = \sum_{n\leqq x} q_k^*(n) = \sum_{n\leqq x} \sum_{d^k\delta=n;(d,\delta)=1} \mu(d) = \sum_{d^k\delta\leqq x;(d,\delta)=1} \mu(d).$$

Let $z=x^{1/k}$ and $0<\rho=\rho(x)<1$, where $\rho(x)$ will be chosen suitably later. If $d^k\delta \leqq x$, then both $d>\rho z$ and $\delta>\rho^{-k}$ cannot simultaneously hold good, and so

$$\begin{aligned}Q_k^*(x) &= \sum_{d^k\delta\leqq x;d\leqq \rho z;(d,\delta)=1} \mu(d) + \sum_{d^k\delta\leqq x;\delta\leqq\rho^{-k};(d,\delta)=1} \mu(d) - \sum_{d\leqq\rho z;\delta\leqq\rho^{-k};(d,\delta)=1} \mu(d) \\ &= S_1 + S_2 - S_3, \quad \text{say}.\end{aligned} \tag{3.2}$$

By (2.1), we have

$$S_1 = \sum_{d \leqq \rho z} \mu(d) \sum_{\delta \leqq x/d^k; (\delta, d)=1} 1 = \sum_{d \leqq \rho z} \mu(d) \varphi\left(\frac{x}{d^k}, d\right)$$

$$= \sum_{d \leqq \rho z} \mu(d) \left\{\frac{x}{d^k} \cdot \frac{\varphi(d)}{d} + O(\tau(d))\right\}$$

$$= x \sum_{n \leqq \rho z} \frac{\mu(n)\varphi(n)}{n^{k+1}} + O\left(\sum_{n \leqq \rho z} \tau(n)\right)$$

$$= x \sum_{n=1}^{\infty} \frac{\mu(n)\varphi(n)}{n^{k+1}} - x \sum_{n > \rho z} \frac{\mu(n)\varphi(n)}{n^{k+1}} + O\left(\sum_{n \leqq \rho z} \tau(n)\right)$$

$$= \alpha_k^* x + O\left(z^k \frac{\delta(\rho z)}{(\rho z)^{k-1}}\right) + O(\rho z \log(\rho z)),$$

by (2.2), (2.12) and $\sum_{n \leqq x} \tau(n) = O(x \log x)$ (cf. [**2**, Theorem 320]). Hence

$$S_1 = \alpha_k^* x + O(\rho^{1-k} z \delta(\rho z)) + O(\rho z \log z). \tag{3.3}$$

We have, by (2.4),

$$S_2 = \sum_{\delta \leqq \rho^{-k}} \sum_{d \leqq (x/\delta)^{1/k}; (d, \delta)=1} \mu(d) = \sum_{\delta \leqq \rho^{-k}} M_\delta\left(\left(\frac{x}{n}\right)^{1/k}\right) = \sum_{n \leqq \rho^{-k}} M_n\left(\left(\frac{x}{n}\right)^{1/k}\right)$$

$$= O\left(\sum_{n \leqq \rho^{-k}} \sigma_{-1+\varepsilon}^*(n) \left(\left(\frac{x}{n}\right)^{1/k}\right) \delta\left(\left(\frac{x}{n}\right)^{1/k}\right)\right).$$

Since $\delta(x)$ is monotonic decreasing and $(x/n)^{1/k} \geqq \rho z$, we have $\delta((x/n)^{1/k}) \leqq \delta(\rho z)$. Hence by (2.3),

$$S_2 = O\left(z \delta(\rho z) \sum_{n \leqq \rho^{-k}} \frac{\sigma_{-1+\varepsilon}^*(n)}{n^{1/k}}\right) = O(\rho^{1-k} z \delta(\rho z)). \tag{3.4}$$

Also, by (2.4) and (2.3), we have

$$S_3 = \sum_{\delta \leqq \rho^{-k}} \sum_{d \leqq \rho z; (d, \delta)=1} \mu(d) = \sum_{\delta \leqq \rho^{-k}} M_\delta(\rho z) = \sum_{n \leqq \rho^{-k}} M_n(\rho z)$$

$$= O\left(\sum_{n \leqq \rho^{-k}} \sigma_{-1+\varepsilon}^*(n) \rho z \delta(\rho z)\right) \tag{3.5}$$

$$= O\left(\rho z \delta(\rho z) \sum_{n \leqq \rho^{-k}} \sigma_{-1+\varepsilon}^*(n)\right) = O(\rho^{1-k} z \delta(\rho z)).$$

Hence, by (3.2), (3.3), (3.4) and (3.5), we have

$$Q_k^*(x) = \alpha_k^* x + O(\rho^{1-k} z \delta(\rho z)) + O(\rho z \log z). \tag{3.6}$$

Now, we choose

$$\rho = \rho(x) = \{\delta(x^{1/2k})\}^{1/k}, \tag{3.7}$$

and write

$$\begin{aligned} f(x) &= \log^{3/5}(x^{1/2k})\{\log\log(x^{1/2k})\}^{-1/5} \\ &= \left(\frac{1}{2k}\right)^{3/5} U^{3/5}(V - \log 2k)^{-1/5}, \end{aligned} \tag{3.8}$$

where $U = \log x$ and $V = \log\log x$.

(3.9) For $V \geqq 2\log 2k$, that is, $U \geqq 4k^2$, $x \geqq \exp(4k^2)$ we have

$$V^{-1/5} \leqq (V - \log 2k)^{-1/5} \leqq (V/2)^{-1/5},$$

and therefore

$$\tfrac{1}{2}k^{-3/5}U^{3/5}V^{-1/5} \leqq f(x) \leqq k^{-3/5}U^{3/5}V^{-1/5}. \tag{3.10}$$

(3.11) We assume without loss of generality that the constant A in (2.5) is less than 1.

By (3.7), (2.5) and (3.8), we have

$$\rho = \exp\{-(A/k)f(x)\}. \tag{3.12}$$

By (3.9), we have $k^{-8/5}U^{3/5}V^{-1/5} \leqq U/2k$.

Hence by (3.10), (3.11), (3.12) and the above,

$$\begin{aligned} \rho &\geqq \exp\{-Ak^{-8/5}U^{3/5}V^{-1/5}\} \geqq \exp\{-k^{-8/5}U^{3/5}V^{-1/5}\} \\ &\geqq \exp\{-U/2k\} = \exp\left\{-\frac{\log x}{2k}\right\}, \end{aligned}$$

so that $\rho \geqq x^{-1/2k}$. Hence $\rho z \geqq x^{1/2k}$. Since $\delta(x)$ is monotonic decreasing, we have by (3.7), $\delta(\rho z) \leqq \delta(x^{1/2k}) = \rho^k$. Hence by (3.10) and (3.12), we have

$$\rho^{1-k}\delta(\rho z) \leqq \rho \leqq \exp\left\{-\frac{A}{2}k^{-8/5}U^{3/5}V^{-1/5}\right\}, \tag{3.13}$$

so that the first O-term in (3.6) is

$$O\left(x^{1/k}\exp\left\{-\frac{A}{2}k^{-8/5}U^{3/5}V^{-1/5}\right\}\right).$$

Also, the second O-term in (3.6) is

$$O\left(x^{1/k}\exp\left\{-\frac{A}{2}k^{-8/5}U^{3/5}V^{-1/5}\right\}\log x\right).$$

Hence, if $\Delta_k^*(x)$ denotes the error term in the asymptotic formula (3.6), then we have

$$\Delta_k^*(x) = O(x^{1/k} \exp\{-B \log^{3/5} x(\log \log x)^{-1/5}\}), \tag{3.14}$$

where B is a constant such that $0<B<(A/2)k^{-8/5}$. Hence Theorem 3.1 follows by (3.6) and (3.14).

THEOREM 3.2. *If the Riemann hypothesis is true, then, for* $x \geqq 3$,

$$Q_k^*(x) = \alpha_k^* x + O(x^{2/(2k+1)}\omega(x)), \tag{3.15}$$

where α_k^* *is given by* (1.2) *and* $\omega(x)$ *is given by* (2.7).

PROOF. Following the same procedure adopted in Theorem 3.1 and making use of (2.6) and (2.13) instead of (2.4) and (2.12), we get the following instead of (3.6):

$$Q_k^*(x) = \alpha_k^* x + O(\rho^{1/2-k} z^{1/2}\omega(\rho z)) + O(\rho z \log z). \tag{3.16}$$

Now, choosing $\rho = z^{-1/(2k+1)}$, we see that $0<\rho<1$ and $\rho^{1/2-k}z^{1/2} = \rho z = x^{2/(2k+1)}$. Since $\omega(x)$ is monotonic increasing, we have $\omega(\rho z) \leqq \omega(z) \leqq \omega(x)$. Also, we see that $\log z = O(\omega(x))$. Hence the first and second O-terms in (3.16) are equal to $O(x^{2/(2k+1)}\omega(x))$. Hence Theorem 3.2 follows.

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