PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 2, February 1973

INNER PRODUCTS CHARACTERIZED BY DIFFERENCE EQUATIONS

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ABSTRACT. A normed linear space X is an inner product space iff, for some integer $k \geqq 3$, $\sum_{t=0}^{k} \binom{k}{t}(-1)^t \|a+ab\|^2=0$ for all a and b in X.

THEOREM. *If X is a linear space with norm $\|\cdot\|$ and, for some integer $k \geqq 3$,*

$$\sum_{t=0}^{k} \binom{k}{t}(-1)^t \|a+tb\|^2 = 0$$

for all $a, b \in X$ then $\sum_{t=0}^{k} \binom{k}{t}(-1)^t \|a+tb\|^2=0$ for every integer $k \geqq 3$ and $\|\cdot\|$ is induced by an inner product on X.

DEFINITION. Suppose X is a normed linear space and k and n are nonnegative integers. Let

$$D_k^n(a, b) = \sum_{t=0}^{k} \binom{k}{t}(-1)^t \|a+(t+n)b\|^2$$

where a and b are in X and $\|\cdot\|$ denotes the norm on X.

PROOF. Suppose k is an integer greater than 2 and $D_k^0(a, b)=0$ for all a and b. Then $D_k^n(a, b)=0$ if n is a nonnegative integer; moreover, $D_k^n(a, b)= D_{k-1}^{n+1}(a, b)-D_{k-1}^n(a, b)$ and hence $D_{k-1}^n(a, b)=D_{k-1}^0(a, b)$.

Suppose m is a positive integer not exceeding k then by iteration we have that

$$D_{k-m}^n(a, b) = \sum_{t=0}^{m-1} \binom{n}{t} D_{k-m+t}^0(a, b) \quad \text{for } n = 0, 1, 2, \cdots, \tag{1}$$

and hence

$$D_0^n(a, b) = \sum_{t=0}^{k-1} \binom{n}{t} D_t^0(a, b) \quad \text{for } n = 0, 1, 2, \cdots. \tag{2}$$

Recall that $D_0^n(a, b)=\|a+nb\|^2$. Hence it follows that

$$\|(1/n)a + b\|^2 = \sum_{t=0}^{k-1} \binom{n}{t} D_t^0(a, b)/n^2. \tag{3}$$

Received by the editors April 20, 1972 and, in revised form, June 21, 1972.
AMS (MOS) subject classifications (1970). Primary 46C10; Secondary 39A25.

We have then that

$$\begin{aligned}\lim_{n\to\infty} \|(1/n)a + b\|^2 &= \|b\|^2 \\ &= \lim_{n\to\infty} \frac{1}{n^2}\sum_{t=0}^{k-1}\binom{n}{t}D_t^0(a,b) \\ &= \lim_{n\to\infty}\Bigg[D_0^0(a,b)/n^2 + \left(\binom{n}{1}\Big/n^2\right)D_1^0(a,b) + \left(\binom{n}{2}\Big/n^2\right)D_2^0(a,b) \\ &\qquad + \left(\binom{n}{3}\Big/n^2\right)D_3^0(a,b) + \cdots + \left(\binom{n}{k-1}\Big/n^2\right)D_{k-1}^0(a,b)\Bigg].\end{aligned}$$

In order that this limit exist it is necessary that $D_l^0(a, b)=0$ if $3\leqq l\leqq k-1$. Hence the limit is $\frac{1}{2}D_2^0(a, b)$ if $k\geqq 3$. Therefore

$$\|b\|^2 = \tfrac{1}{2}[\|a + 2b\|^2 - 2\|a + ab\|^2 + \|a\|^2]$$

for all a and b in X, which is a simple reformulation of the parallelogram law.

Hence the space is an inner product space from which it is easy to establish that $D_k^0(a, b)=0$ if k is any integer greater than 2.

Reference

M. M. Day, *Normed linear spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N.F., Heft 21, Academic Press, New York; Springer-Verlag, Berlin, 1962. MR **26** #2847.

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