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ON THE EXISTENCE OF REGULAR APPROXIMATE DIFFERENTIALS

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ABSTRACT. We prove that for continuous real-valued functions on an open set in n-space, a sufficient condition for the existence a.e. of a regular approximate differential is that the functions have an ordinary total differential a.e. with respect to all but one variable.

1. **Introduction.** T. Radó ([4], [5]) and R. Caccioppoli and Scorza Dragoni [1] introduced the concept of regular approximate differential (Definition 3, below) and proved that in order for a continuous real-valued function defined on a planar open set to have a regular approximate differential a.e., it suffices that the function have first partial derivatives a.e. In this note we extend this result in a natural way to n-space by proving that, again for a continuous function on an open set, a sufficient condition for the existence a.e. of a regular approximate differential is the existence a.e. of a total differential with respect to all but one variable. This strengthens a result of Lodigiani [2], who required that the function have partial derivatives everywhere which are continuous with respect to all but one variable.

2. **Notation, definitions and basic theorems.** Let $p=(p^1, \cdots, p^n)$ denote points in n-space R^n, and $\|p-q\|=[\sum_{j=1}^n (p^j-q^j)^2]^{1/2}$ the distance between p and q. We use $\nabla f(p)$ for the n-tuple of partial derivatives $(f'_1(p), \cdots, f'_n(p))$ and set

$$\nabla f(p) \cdot (p - q) = \sum_{j=1}^{n} f'_j(p)(p^j - q^j).$$

The notation mS will denote n-dimensional Lebesgue measure of a set S in R^n.

DEFINITION 1. Given a real-valued function $f: S \to R$ on a subset S of R^n, suppose f has partial derivatives at a point p in S. Let

$$e(p, q) = \frac{f(q) - f(p) - \nabla f(p) \cdot (q - p)}{\|q - p\|}, \qquad p \neq q.$$

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Then f is said to have a *total differential* (in the Stolz sense) at p if $\lim_{q\to p} e(p, q)=0$.

DEFINITION 2. Given any point p in R^n let $\mathscr{C}(p)$ denote any family of oriented n-cubes (edges parallel to the coordinate axes) such that (i) p is the center of each n-cube of $\mathscr{C}(p)$, and (ii) p is a point of density for $\bigcup$ fr C, $C\in\mathscr{C}(p)$, where fr C denotes the frontier of C. We term $\mathscr{C}(p)$ a *thick regular family* of n-cubes at p.

The following lemma is easily verified.

LEMMA 1. *If p is a point of (linear) density of a subset S of R^n in the direction of every coordinate axis, then there exists a thick regular family of n-cubes $\mathscr{C}(p)$ such that the lines through p parallel to the coordinate axes intersect the faces of each n-cube in a point of S.*

DEFINITION 3. Given a real-valued function $f: S\to R$, where S is a subset of R^n, we say that f has a *regular approximate differential* at a point p in S if there exists a thick regular family $\mathscr{C}(p)$ such that f restricted to $\bigcup (S\cap \mathrm{fr} C)$, $C\in\mathscr{C}(p)$, has a total differential at p.

LEMMA 2. *Let p be a point of density of a measurable set S in R^n. Then for every $\eta>0$ there exists a $\delta>0$ such that whenever $\|p-q\|<\delta$ for a point q in R^n there corresponds a point q_* in S satisfying the inequality $\|q-q_*\|<\eta\|p-q\|$.*

PROOF. This is a simple exercise in the definition of density, and may be found implicitly in Rademacher [3].

COROLLARY. *Let S be a bounded measurable set in R^n, and let $\eta>0$ be given. Then for every $\varepsilon>0$ there corresponds a closed subset F of S and a number $\delta>0$ for which the following two properties hold:*

(i) *$m(S-F)<\varepsilon$.*

(ii) *For every p in F, q in R^n with $\|p-q\|<\delta$ and $p^k=q^k$ for some integer k, there corresponds a point q_* in S such that $q_*^k=q^k$ and $\|q_*-q\|<\eta\|p-q\|$.*

PROOF. This follows easily from Lemma 2 by applying it to $(n-1)$-dimensional sections of S and choosing the set F to uniformize the δ.

LEMMA 3. *Let $f: S\to R$ be a continuous real-valued function defined on a bounded open set S with the following property: for almost every point $p\in S$ there exist $\delta>0$, $M>0$ such that*

(*) *$|f(p)-f(q)|\leqq M\|p-q\|$ whenever $q\in S$, $\|p-q\|<\delta$ and $p^k=q^k$ for some k.*

Then for every $\varepsilon>0$ there exists a measurable subset E of S, and numbers $\delta>0$, $M>0$ such that (1) $m(S-E)<\varepsilon$, (2) for every $p\in E$ condition (*) in the hypothesis holds.

PROOF. Define the set E_m to consist of all points p in S such that $|f(p)-f(q)|\leqq m\|p-q\|$ if $q\in S$, $\|p-q\|<1/m$ and $p^k=q^k$ for some k. Clearly, E_m is an ascending sequence of sets with $m(S-\bigcup_m E_m)=0$ in view of the hypothesis. To show that each E_m is closed relative to S let p_n denote any sequence of points of E_m converging to $p_0\in S$. Suppose $q\in S$, $q^k=p^k$, and $\|p_0-q\|<1/m$. Then, for n large enough, $\|p_n-q\|<1/m$. Let $\bar{p}_n$ denote the point such that $\bar{p}_n^k=q^k=p^k$, but $\bar{p}_n^j=p_n^j$ for $j\neq k$. Then, for n large enough, $\bar{p}_n\in S$ since S is open, and so $|f(\bar{p}_n)-f(q)|\leqq m\|\bar{p}_n-q\|$. Since f is continuous on S and clearly $\bar{p}_n\to p_0$ we infer that

$$|f(p_0)-f(q)|\leqq m\|p_0-q\|$$

and so $p_0\in E_m$, and the proof that E_m is closed relative to S is complete. Thus for any assigned $\varepsilon>0$ there is an m_0 such that $m(S-E_{m_0})<\varepsilon$. We therefore take $E=E_{m_0}$, $\delta=1/m_0$ and $M=m_0$, and the proof is complete.

3. **Main theorem.**

THEOREM. *Let $f:S\to R$ be a continuous real-valued function defined on an open bounded set S in R^n. Assume that f has a total differential a.e. in the direction of every $(n-1)$-dimensional coordinate hyperplane. Then f has a regular approximate differential a.e. on S.*

PROOF. Let $\varepsilon>0$, $\gamma>0$ be given arbitrarily. In view of the hypotheses of the theorem, the conditions of Lemma 3 are satisfied. Select a set $E=E_1$ and numbers $\delta=\delta_1$ and $M=M_1$, satisfying the conclusion of Lemma 3. By hypothesis the partial derivatives exist a.e. on S and hence by the well-known theorem of Stepanoff [7], there exists a measurable subset E_2 of E_1 such that f has a total differential at each point of E_2 with respect to E_2 and $m(E_1-E_2)<\varepsilon$. By Lusin's theorem we may assume without loss that E_2 is closed and the partial derivatives are continuous on E_2, hence $|\nabla f(p)|\leqq M_2$ for some constant M_2 and every $p\in E_2$. Further, let E_3 denote the set of all points of E_2 which are points of density of E_1 in the direction of every coordinate axis and in the direction of every $(n-1)$-dimensional coordinate hyperplane. Then $m(E_2-E_3)=0$ (see Saks [**6**, p. 298] for the linear case). Finally, let E_4 be (see corollary of Lemma 2) a measurable subset of E_3 and let $\delta_2>0$ chosen so that

(i) $m(E_3-E_4)<\varepsilon$,

(ii) for every $p\in E_4$ and $q\in S$ with $\|p-q\|<\delta_2$ and $p^k=q^k$ for some k, there corresponds a $q_*\in E_1$ such that

$$\|q-q_*\|<\frac{\gamma}{3M}\|p-q\| \;\&\; q_*^k=q^k \tag{1}$$

where $M=\max(M_1, M_2)$.

Consider now any point $p_0 \in E_4$, and let $\delta_3 > 0$ be chosen so that

$$|f(\bar{q}) - f(p_0) - \nabla f(p_0) \cdot (\bar{q} - p_0)| \leqq (\gamma/6) \|\bar{q} - p_0\| \quad \text{when } \|\bar{q} - p_0\| < \delta_3, \ \bar{q} \in E_2. \tag{2}$$

Take now any point $p_0 \in E_4$. Let $\mathscr{C}(p_0)$ denote a thick regular family of n-cubes centered at p_0 such that (see Lemma 1) the lines through p_0 parallel to the coordinate axes intersect the frontiers of the n-cubes in points of E_3. Let $\delta = \min(\delta_1, \delta_2, \delta_3)$ and take any point $q \in S$ on the frontier of any n-cube C of $\mathscr{C}(p_0)$ having diagonal-length less than δ. Suppose without loss of generality that the 1-axis is perpendicular to the face whereon q lies. Then $q = (p_0^1 + h, q^2, q^3, \cdots, q^n)$ for some number h. Let $p_1 = (p_0^1 + h, p_0^2, \cdots, p_0^n)$. Then $p_1 \in E_3$, so that since $\|q - p_1\| < \delta_2$ we have a $q_* = (p_0^1 + h, q_*^2, \cdots, q_*^n) \in E_1$ satisfying (1). Then $\|q - q_*\| < \delta_1$ and $|q_* - p_0| < \delta_3$, and so we have the following sequences of inequalities:

$$\begin{aligned}
&|f(q) - f(p_0) - \nabla f(p_0) \cdot (q - p_0)| \\
&\quad \leqq |f(q) - f(q_*)| + |f(q_*) - f(p_0) - \nabla f(p_0) \cdot (q - p_0)| \\
&\quad \leqq M_1 \|q - q_*\| + |f(q_*) - f(p_0) - \nabla f(p_0) \cdot (q_* - p_0)| \\
&\qquad + |\nabla f(p)| \cdot \|q - q_*\| \\
&\quad \leqq M_1(\gamma/3M_1) \|q - p_1\| + (\gamma/6) \|q_* - p_0\| + M_2(\gamma/3M_2) \|q - p_1\| \\
&\quad \leqq (\gamma/3) \|q - p_0\| + (\gamma/3) \|q - p_0\| + (\gamma/3) \|q - p_0\| = \gamma \|q - p_0\|.
\end{aligned}$$

Thus, f has a total differential at p_0 relative to the part of S in $\bigcup$ fr C, $C \in \mathscr{C}(p_0)$, and so f has a regular approximate differential at p_0. Since p_0 is an arbitrary point of E_4 and $m(S - E_4) < 3\varepsilon$, the theorem is proved.

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