PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 2, February 1973

ON COMPOSITE LOOP FUNCTORS[1]

K. A. HARDIE

ABSTRACT. P is a space with two points in a certain convenient category $\boldsymbol{CG}$ of pointed topological spaces. If $T:\boldsymbol{CG}\to\boldsymbol{CG}$ is a P-functor and $X\in\boldsymbol{CG}$, we establish a homotopy equivalence $\Omega TX\simeq\Omega T*\times\Omega(X\wedge F)$, where F is the fibre of $T(*):TP\to T*$.

Let $\boldsymbol{CG}$ denote the category of compactly generated Hausdorff topological spaces with base points (denoted by $*$) such that, for each $X\in\boldsymbol{CG}$, $(X, *)$ has the homotopy extension property. Let Ω denote the loop space functor for pointed topological spaces. In a recent paper Gray [2] has proved the homotopy equivalence

$$\Omega(X\vee Y)\simeq\Omega Y\times\Omega(X\times\Omega Y/\Omega Y). \tag{1}$$

The purpose of this note is to obtain a similar decomposition for ΩTX, where $T:\boldsymbol{CG}\to\boldsymbol{CG}$ is a P-functor [3], [4].

Let (W, V) be an NDR pair in the sense of [7] and suppose that there exists a retraction $\phi: W\to V$. For each $X\in\boldsymbol{CG}$, let $TX=T_\phi X$ be the space obtained from $X\times W$ (i.e. the $\boldsymbol{CG}$ product) by performing the identification

$$(*, w)=(x, \phi w)\qquad(w\in W, x\in X). \tag{2}$$

Given $f:X\to Y$, let $Tf\{(x, w)\}=\{(fx, w)\}$. Then we have the following

THEOREM. *$T=T_\phi:\boldsymbol{CG}\to\boldsymbol{CG}$ is a functor and $\Omega TX\simeq\Omega T*\times\Omega(X\wedge F)$, for each $X\in\boldsymbol{CG}$, where F is the fibre of ϕ.*

Let $P\in\boldsymbol{CG}$ be a space with two points. There is a retraction k of P onto its base point. Hence if $S:\boldsymbol{CG}\to\boldsymbol{CG}$ is a functor, Sk is a retraction of SP onto a subspace isomorphic with $S*$. S is a *P-functor* if S is naturally equivalent to T_ϕ, for $\phi=Sk$. (The sense differs slightly from that of [3], [4].) For example if $TX=X\vee Y$, $Tf=f\vee i_Y$ (for a fixed $Y\in\boldsymbol{CG}, f:X\to X'$) then T is a P-functor and applying the theorem we recover (1), for $F=(\Omega Y)^+=P\times\Omega Y/\Omega Y$, as is observed in the proof of [2, Lemma 3]. Let Σ

Received by the editors February 25, 1972.
AMS (MOS) subject classifications (1970). Primary 55D35, 55D10, 55D05, 55E20.
Key words and phrases. Loop space, homotopy equivalence, cofibration, wedge.

[1] Prepared with the assistance of South African Council for Scientific and Industrial Research grant 40-332.

denote the suspension functor. Then as another application we shall obtain the following result.

COROLLARY. $\Omega(X \times Y/Y) \simeq \Omega X \times \Omega(Y \wedge \Sigma\Omega X)$.

Combining the Corollary with (1) yields

$$\Omega(X \vee Y) \simeq \Omega Y \times \Omega X \times \Omega((\Omega Y) \wedge \Sigma\Omega X), \tag{3}$$

which is consistent with and may be regarded as a nonweak form of [**5**, 3.7, p. 281]:

$$\Omega(X \vee Y) \simeq (\text{weak})\, \Omega X \times \Omega Y \times \Omega(X \flat Y).$$

Before proceeding to the proofs of the Theorem and Corollary we remark that the form of the Serre-Cartan construction appropriate to ***CG*** uses the space EX of Moore paths in X [**1**] with compactly generated topology. (In this connection I wish to acknowledge a helpful conversation with Professor Eldon Dyer.) We recall that a Moore path is a pair (f, r), where r is a nonnegative real number and f a map of the closed interval $[0, r]$ into X. There is a map $\lambda: X \to EX$ given by $\lambda x = (x, 0)$ which is pair-homotopy equivalent to the identity $X \to X$. It follows that λ has the weak homotopy extension property. Moreover if we set $\mu(f, r) = \min(r, 1)$ we obtain a map $\mu: EX \to I$ with the property that $\mu^{-1}(0) = \lambda(X)$. Hence, by [**6**, Satz I], λ is a cofibration. If $X \in \boldsymbol{CG}$, it follows that $EX \in \boldsymbol{CG}$. Similarly let $\Omega' X$ be the space of Moore loops on X. Since $\Omega' X$ has the weak homotopy extension property [**9**, Satz, p. 180], a second application of [**6**, Satz I] shows that $\Omega' X \in \boldsymbol{CG}$. If F is the (Moore-path) fibre of $f: X \to Y$ we have a diagram

$$\begin{array}{ccccccc}
* \to \Omega' Y & \to & F & = & F & \to & LY \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
* & \to & X & & * & \to & Y
\end{array}$$

of pullback rectangles and hence by [**8**, Theorem 12] the morphisms on the top row are cofibrations. Thus $F \in \boldsymbol{CG}$ and $LY \in \boldsymbol{CG}$.

PROOF OF THEOREM. Let $\psi = \psi X$ denote the identification map associated with (2). Then we have a diagram

$$\begin{array}{ccccc}
X \times V \cup * \times W & \xrightarrow{\subseteq} & X \times W & \xrightarrow{* \times i_W} & * \times W \\
\Big\downarrow{\scriptstyle \phi \cdot \text{projection}} & & \Big\downarrow{\scriptstyle \psi X} & & \Big\downarrow{\scriptstyle \psi *} \\
V & \longrightarrow & TX & \xrightarrow{q} & T*
\end{array}$$

in which the composite of the bottom row is an equivalence and the left-hand rectangle is a pushout in the category of pointed topological spaces.

Moreover an application of [7, Lemma 8.5] shows that (TX, V) is an NDR pair. Since $(V, *)$ is NDR, [7, Lemma 7.2] implies that $(TX, *)$ is NDR and hence $TX \in \boldsymbol{CG}$. Pulling back the Moore path fibration $LT* \to T*$ (with contractible total space) over the diagram, we obtain an upper diagram

$$\begin{array}{ccccc} (X \times LV) \cup (* \times F) & \longrightarrow & X \times F & \longrightarrow & F \\ \downarrow & & \downarrow & & \downarrow \\ LV & \longrightarrow & G & \longrightarrow & LT* \end{array}$$

in which G is the fibre of q and the left-hand rectangle is again a pushout. (If in $\boldsymbol{CG}$ a map is pulled back over the pushout of a cofibration then the upper diagram is necessarily a pushout.) Since (G, LV) and $(X \times F, X \times LV \cup * \times F)$ are NDR by [**8**, Theorem 12], we have

$$\begin{aligned} G \simeq G/LV &\approx X \times F/(X \times LV \cup * \times F) \\ &\simeq X \times F/(X \times * \cup * \times F) = X \wedge F. \end{aligned}$$

But q is a retraction and hence its Serre-Cartan fibration has a cross section. As in [**2**, Lemma 2] we may obtain $\Omega' TX \simeq \Omega' G \times \Omega' T*$, completing the proof.

Proof of Corollary. For a fixed $X \in \boldsymbol{CG}$, let $TY = X \times Y/Y$. Certainly T is a P-functor. Moreover ϕ is the folding map $X \vee X \to X$ and it is easily shown that $F \simeq \Sigma\Omega X$.

References

1. J. F. Adams and P. J. Hilton, *On the chain algebra of a loop space*, Comment. Math. Helv. **30** (1956), 305–330. MR **17**, 1119.

2. B. Gray, *A note on the Hilton-Milnor theorem*, Topology **10** (1971), 199–201. MR **43** #6921.

3. K. A. Hardie, *Weak homotopy equivalence of P-functors*, Quart. J. Math. Oxford Ser. (2) **19** (1968), 17–31. MR **37** #3564.

4. ———, *Homotopy of natural transformations*, Canad. J. Math. **22** (1970), 332–341. MR **41** #6209.

5. P. J. Hilton, *Homotopy theory of modules and duality*, Symposium Internacional de Topologia Algebraica, Universidad Nacional Autonoma de Mexico and UNESCO, Mexico City, 1958, pp. 273–281. MR **20** #4588.

6. D. Puppe, *Bemerkungen über die Erweiterung von Homotopien*, Arch. Math. (Basel) **18** (1967), 81–88. MR **34** #6770.

7. N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. **14** (1967), 133–152. MR **35** #970.

8. A. Strøm, *Note on cofibrations*. II, Math. Scand. **22** (1968), 130–142. MR **39** #4846.

9. T. tom Dieck, K. H. Kamps and D. Puppe, *Homotopietheorie*, Lecture Notes in Math., vol. 157, Springer-Verlag, Berlin and New York, 1970.

Department of Mathematics, University of Cape Town, Rondebosch, C.P., South Africa