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SOME ALGEBRAIC K-THEORETIC APPLICATIONS OF THE LF AND NF FUNCTORS

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ABSTRACT. Using previous results of H. Bass, we compute $\mathrm{Pic}(P^1(R))$ and $L^nN^iK_0(\Lambda[G])$ where R is a commutative ring, Λ any commutative finite algebra over a Dedekind ring, and G any finitely generated free abelian group or monoid.

Introduction. This paper is a sequel to some results on LF and NF functors introduced by Bass in [1]. The notations are those of [1]. In §1, some results are given on Picard group of the projective line and §2 deals with $L^nN^iK_0(\Lambda)$ and $L^nN^iK_0(\Lambda[G])$ for Λ any commutative finite algebra over a Dedekind domain and G a free abelian group or monoid. Lastly we observe that $L^nN^iK_0(A)$ is a filtered $K_0(R)$-module if A is an algebra over a commutative ring R.

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1. Let R be a commutative ring, $\mathbf{Pic}(R)$ the category (with product $\otimes$) of finitely generated projective modules of rank 1 and $(T, T_\pm)$ an oriented cycle. We write $\mathbf{Pic}(P^1(R))$ for the fibre product category

$$\mathbf{Pic}(R[T_+]) \underset{\mathrm{Pic}(R[T])}{\times} \mathbf{Pic}(R[T_-])$$

and denote $K_0(\mathbf{Pic}(P^1(R)))$ by $\mathrm{Pic}(P^1(R))$.

THEOREM 1.1. *For* $P \in \mathbf{Pic}(R[T_+])$, $P_0 = P \otimes_{R[T_+]} R \in \mathbf{Pic}(R)$; *let* $P \approx P_0[T_+]$. *Then* $\mathrm{Pic}(P^1(R)) \approx H_0(R) \oplus \mathrm{Pic}(R)$.

PROOF. From [1, p. 365, Theorem 4.3], we obtain an exact sequence

$$\text{(I)}\quad \begin{aligned} &K_1(\mathbf{Pic}(P^1(R))) \to U(R[T_+]) \oplus U(R[T_-]) \to U(R[T]) \\ &\qquad \to \mathrm{Pic}(P^1(R)) \to \mathrm{Pic}(R[T_+]) \oplus \mathrm{Pic}\ R[T_-] \to \mathrm{Pic}(R[T]), \end{aligned}$$

since $K_1(\mathbf{Pic}(R)) \approx U(R)$ and $K_0(\mathbf{Pic}(R)) \approx \mathrm{Pic}(R)$.

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Also by [**1**, p. 670, Corollary 7.7] and the fact that $LU \approx H_0$ (see [**1**, p. 671, Proposition 7.8]) we obtain from (I) the following exact sequence

$$\text{(II)} \qquad 0 \to H_0(R) \xrightarrow{i} \operatorname{Pic}(P^1(R)) \xrightarrow{j} \operatorname{Pic}(R) \to 0.$$

Now define $\eta:\operatorname{Pic}(R)\to\operatorname{Pic}(P^1(R))$ by $\eta[P]=(P[T_+], 1_{P[T]}, P[T_-])$. So $j\eta[P]=j(P[T_+], 1_{P[T]}, P[T_-])=[P[T_+]\otimes_{R[T_+]} R]=[P]$. So the sequence (II) is split exact and hence $\operatorname{Pic}(P^1(R))\approx\operatorname{Pic}(R)\oplus H_0(R)$.

COROLLARY 1.2. *If R is a commutative-Noetherian ring of stable Serre dimension* $\leqq 1$ *then* $\operatorname{Pic}(P^1(R))\approx K_0(R)$.

PROOF. Follows from $K_0(R)\approx H_0(R)\oplus Rk_0(R)$ and the fact that $Rk_0(R)\approx\operatorname{Pic}(R)$ if and only if stable Serre dimension $R\leqq 1$ ([**2**, p. 59]).

COROLLARY 1.3. *Suppose R is a commutative Artinian ring. Then* $\operatorname{Pic}(P^1(R))\approx K_0'(P^1(R))\approx H_0(R)$. *When the cartesian square*

$$\begin{array}{ccc} P(P^1(R)) & \longrightarrow & P(R[T_-]) \\ \downarrow & & \downarrow \\ P(R[T_+]) & \longrightarrow & P(R[T]) \end{array}$$

is E-surjective, then $\operatorname{Pic}(P^1(R))\approx K_0(P^1(R))$.

PROOF. Since $\tau_\pm: R[T_+]\to R[T]$ are inclusions, $H_0(R[T_\pm])\to H_0(R[T])$ are injective and we can replace the K_0's in the exact sequence for $K_0'(P^1(R))$ in [**1**, p. 679], by Rk_0's (see [**1**, p. 466]). The resulting exact sequence is then mapped into (I) in the proof of 1.1 and the result follows by applying 1.1 and Lemma 7.6 of [**2**].

2. Let $\boldsymbol{R}$ be the category of rings (with unit) and $\boldsymbol{Ab}$ the category of Abelian groups.

LEMMA 2.1. *If a functor $F:\boldsymbol{R}\to\boldsymbol{Ab}$ has the property that $F(R)\to F(R/N)$ is an isomorphism when N is a nilpotent ideal of R, then LF, NF have the same property. Hence $L^nN^iK_0$ has this property.*

Proof is easy and is omitted.

LEMMA 2.2 ([**1**, p. 163]). *Let R be a Dedekind ring with quotient field L. Suppose Λ is a finite R-algebra. Then there is a largest two-sided nilpotent ideal N in Λ. If Γ is the R-torsion submodule of Λ/N, then Γ is a semisimple ring and $\Lambda/N\approx\Gamma\times A$ where A is an R-order in a semisimple algebra.*

THEOREM 2.3. *Let R be a Dedekind ring with quotient field L, Λ any commutative finite R-algebra, Γ, A as in 2.2, G a finitely generated free* *abelian group or monoid. Then*

(i) $L^nN^iK_0(\Lambda)\approx L^nN^iK_0(\Lambda[G])=0$ *for* $n>0$ *and* $i>0$, *or for* $n>1$ *and* $i\geqq 0$,

(ii) $\det_0(\Lambda[G]):RK_0(\Lambda[G])\to \mathrm{Pic}(\Lambda[G])$ *is an isomorphism*,

(iii) $LK_0(\Lambda)\approx LK_0(\Lambda[G])\approx LK_0(A)$ *is a torsionfree abelian group*,

(iv) $K_0(\Lambda[G])\approx H_0(\Gamma)\oplus K_0(A[G])$,

(v) $N^iK_0(\Lambda)\approx N^iK_0(A)$.

Similarly $N^iK_0(\Lambda[G])\approx N^iK_0(A[G])$.

PROOF. By 2.1 we have $L^nN^iK_0(\Lambda)\approx L^nN^iK_0(\Lambda/N)$. Also

$$L^nN^iK_0(\Lambda[G]) \approx L^nN^iK_0(\Lambda/N[G])$$

from (2.1) and Grothendieck's theorem [**1**, p. 636]. Since $\Lambda/N=\Gamma\times A$, the above theorem reduces to $\Lambda=\Gamma$ and $\Lambda=A$.

So (i) follows from [**1**, p. 688, Theorem 10.2]; (ii) follows from [**1**, p. 690, Theorem 10.4]; (iii) follows from [**1**, p. 690, 10.4(c)], Grothendieck's theorem, and [**2**, Lemma 7.6]; (iv) follows from Grothendieck's theorem, and $Rk_0(\Gamma)=\mathrm{Pic}(\Gamma)=0$, Γ being semisimple; (v) follows from [**1**, p. 685, Theorem 10.1].

COROLLARY 2.4. *Suppose* R, Λ, Γ *are as in* 2.3, *then* $\mathrm{Pic}(P^1(\Lambda))\approx K_0(\Lambda)$. *If* $R=Z$ *or* $F[t]$, *the polynomial ring in* t *over a finite field, then* $\mathrm{Pic}(P^1(\Lambda))$ *is a finitely generated abelian group.*

PROOF. Follows from the union of 1.1, 2.3 and [**1**, p. 545, Theorem 2.7].

3. Let A be an algebra over a commutative ring R. In [**1**, p. 473], Bass defined a $K_0(R)$-module filtration $F^i_RK_0(A)$ on $K_0(A)$ using the space $\max(R)$ of maximal ideals of R.

We now observe the following:

3.1. If a functor F on R-algebras has a natural filtration, so do LF and NF. So, if F is a filtered $K_0(R)$-module so are LF and NF. Hence $L^nN^iK_0(A)$ is a filtered $K_0(R)$-module.

Proof is easy and is omitted.

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