PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 2, February 1973

# WEAKLY COMPACT POSITIVE OPERATORS ON SUMMABLE FUNCTIONS

PAUL C. SHIELDS

ABSTRACT. This paper uses elementary integration and operator techniques combined with some theorems of Nakano to give a simple proof of the well-known Dunford-Pettis theorem for positive operators.

The Dunford-Pettis theorem [1] states that a weakly compact linear operator on $L^1$ which has separable range maps weakly convergent sequences into norm convergent sequences. We prove

THEOREM 1. *If $T$ is a weakly compact positive linear operator on $L^1(S, \Sigma, \mu)$ then $T$ maps weakly convergent sequences into norm convergent sequences.*

For simplicity let us assume that $\mu$ is a finite measure. The space $L^1(S, \Sigma, \mu)$ is a vector lattice where $f \geqq 0$ is to mean that $f(s) \geqq 0$ a.e. The norm-bounded operators on $L^1(S, \Sigma, \mu)$ also form a vector lattice where $T \geqq 0$ means that $Tf \geqq 0$ if $f \geqq 0$ and $T \vee S$ is defined for $f \geqq 0$ by

$$(T \vee S)f = \sup_{0 \leqq g \leqq f} Tg + S(f - g).$$

Let $B(L^1)$ denote the set of order-continuous operators on $L^1$, that is, $T \in B(L^1)$ if and only if $T$ maps dominated a.e. convergent sequences into dominated a.e. convergent sequences. It is fairly easy to see that $B(L^1)$ consists of all the norm-bounded operators on $L^1$. Except for this latter result, all the above definitions and results hold for $L^\infty$. If $L^\infty$ is not finite-dimensional then $B(L^\infty)$ is a proper lattice subspace of the set of all norm-bounded operators on $L^\infty$.

Let $L$ denote either of the two spaces $L^1$ or $L^\infty$ and for $N \subseteq B(L)$; let

$$N^\perp = \{T \mid |T| \wedge |C| = 0 \text{ for all } C \in N\}.$$

Suppose $\varphi$ is the operator defined by $\varphi f = \int f\, d\mu$, $f \in L$, and that $F$ is the set of operators in $B(L)$ of finite rank. It is easy to see that

(i) $T \in F^{\perp\perp}$ *if and only if* $|T| = \sup_n |T| \wedge n\varphi$.

Received by the editors March 16, 1971.

*AMS (MOS) subject classifications* (1970). Primary 47B55, 46A40.

*Key words and phrases.* Weakly compact operator, positive operator, vector lattice.

Nakano [3, Theorem 5.2] gives the following characterization of $F^{\perp\perp}$.

(ii) *$T \in F^{\perp\perp}$ if and only if $T$ maps order-bounded sequences which converge in measure into order-bounded sequences which converge a.e.*

A second result of Nakano [3, Theorem 5.3] is the following:

(iii) *If $T \in B(L)$ is dominated by an operator in $F$, then $T$ maps weakly convergent sequences into dominated sequences which converge a.e.*

It will now be shown that Theorem 1 is a consequence of (iii). To do this put

$$\lambda(A) = T^*(\chi_A), \qquad A \in \Sigma,$$

where $T^*$ is the adjoint of the given positive weakly compact operator $T$ on $L^1(S, \Sigma, \mu)$. Since $T^{**}$ maps $L^{1**}$ into $L^1$ it follows that $\lambda$ is weakly countably additive and hence countably additive and absolutely continuous with respect to $\mu$ [2, Theorem IV. 10.1]. Thus we can write

$$T^*f = \int f \, d\lambda, \qquad f \in L^\infty,$$

and obtain

(iv) *If $0 \leqq f_n \leqq f$ and $\{f_n\}$ converges in measure to 0, then $\{T^*f_n\}$ converges to 0 a.e.*

The proof of (iv) is quite similar to the analogous scalar measure result. For any given $\varepsilon > 0$, put $E_n = \{s | f_n(s) \geqq \varepsilon\}$ so that

$$\begin{aligned} T^*f_n &= \int_{E_n} f_n \, d\lambda + \int_{E_n^c} f_n \, d\lambda \\ &\leqq \lambda(E_n) \, \|f\|_\infty + \varepsilon\lambda(S). \end{aligned}$$

Since $\lambda$ is absolutely continuous with respect to $\mu$ it follows that $\|\lambda(E_n)\|_\infty \to 0$ and hence that $\lim \lambda(E_n) = 0$ a.e.

Now apply the result (ii) to $T^*$ to conclude that $T^* \in F^{\perp\perp}$ and hence that $T^* = \sup_n (T^*) \wedge n\varphi$. From this it follows easily that $T = \sup_n T \wedge n\varphi$.

This establishes the result

(v) *If $T$ is a weakly compact positive operator on $L^1$, then there is a sequence $\{T_n\}$ of positive operators which increases to $T$ such that each $T_n$ is dominated by an operator of finite rank.*

Suppose $\{a_m\}$ converges to 0 weakly. Then $Ta_m = T_n a_m + (T - T_n)a_m$ so that

$$\text{(vi)} \qquad \int |Ta_m| \, d\mu \leqq \int |T_n a_m| \, d\mu + \int (T - T_n) \, |a_m| \, d\mu.$$

Note that $\{|a_m|\}$ is weakly compact, while $\{(T^* - T_n^*)1\}$ decreases to 0, so that

$$\lim_n \int (T - T_n) \, |a_m| \, d\mu = 0$$

uniformly in $m$. Furthermore, if $n$ is fixed, the result (iii) gives

$$\lim_m \int |T_n a_m| \, d\mu = 0.$$

These two facts combine with (vi) to show that $\lim_m \int |Ta_m| \, d\mu = 0$, which completes the proof of Theorem 1.

REMARK. If one can show that

$$|T|^{**} = |T^{**}| \quad \textit{for } T \in B(L^1), \tag{vii}$$

it is easy to remove the assumption that $T$ be positive in Theorem 1. The result (vii), however, seems very difficult to prove.

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*Current address*: Department of Mathematics, Stanford University, Stanford, California 94305