PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 2, February 1973

AN ANALOGUE OF L'HOSPITAL'S RULE[1]

MAXWELL ROSENLICHT

ABSTRACT. Formal power series expansions have proved as useful in differential algebra as in other fields of mathematics. The present work magnifies one small aspect of their theory (leading terms) to get a simple result that is independent of the expansions themselves and still of considerable utility.

For the convenience of the reader we include a proof of the following known fact.

LEMMA. *Let k be a subfield of the field K, which is complete with respect to a real discrete k-place P whose residue field is separably algebraic over k. Then the map induced by P from the algebraic closure of k in K into the residue field is surjective.*

The words "real discrete" (or "rank one discrete") mean that P is associated with a valuation ord_P, trivial on k, whose value group is $\mathbf{Z}$. Since P induces an embedding of the algebraic closure of k in K into the residue field, we may assume that k is algebraically closed in K and then have to prove that k is a system of representatives for the residue field. For some arbitrary fixed α in the residue field, let $f(X) \in k[X]$ be the minimal polynomial of α over k, so that $f(\alpha)=0, f'(\alpha)\neq 0$. Choose $a_0 \in K$ to be a representative for α, so that $\mathrm{ord}_P f(a_0)>0$, $\mathrm{ord}_P f'(a_0)=0$. We can define $a_1, a_2, \cdots$ inductively by setting $a_{n+1}=a_n-f(a_n)/f'(a_n)$ for $n=0, 1, \cdots$, since for each n we get $\mathrm{ord}_P f'(a_n)=0$. For any $a \in K$ we have $f(X)= f(a)+f'(a)(X-a)+(\text{element of } k[a, X])(X-a)^2$, and we prove inductively that for each $n=0, 1, \cdots$ we have $\mathrm{ord}_P f(a_n)\geqq 2^n$, $\mathrm{ord}_P(a_{n+1}-a_n)\geqq 2^n$. Thus $\lim_{n\to\infty} a_n$ is both a representative of α and a zero of $f(X)$, hence an element of k.

The following is our main result.

THEOREM. *Let K be a field of characteristic zero, k a subfield of K, P a real discrete k-place of K whose residue field is algebraic over k, D a derivation of K that is continuous in the topology of P and that maps k into itself. Let x, y be nonzero elements of K such that each of $x(P)$, $y(P)$ is*

Received by the editors April 20, 1972.
AMS (MOS) subject classifications. Primary 12H05.
[1] Research supported by National Science Foundation grant number GP-20532A.

either 0 *or* ∞. *Then*

(1) *if* $\operatorname{ord}_P(Dx/x) \geqq 0$, *then* $\operatorname{ord}_P(Dy/y) \geqq 0$. *Here D induces a derivation on the residue field of P. Denoting this residue field derivation by the same symbol D, for any* $z \in K$ *such that* $\operatorname{ord}_P z \geqq 0$ *we have* $(Dz)(P) = D(z(P))$;

(2) *if* $\operatorname{ord}_P(Dx/x) < 0$, *then* $\operatorname{ord}_P(Dx/x) = \operatorname{ord}_P(Dy/y)$ *and therefore* $\operatorname{ord}_P(y/x) = \operatorname{ord}_P(Dy/Dx)$. *In addition*

$$\left(\frac{y}{x}\right)(P) = \left(\frac{Dy}{Dx}\right)(P) \qquad (\textit{L'Hospital's rule}).$$

Note that the P-image of an element $z \in K$ is here given the geometric notation $z(P)$. For the proof, note first that since D is continuous in the topology of P it extends uniquely to a continuous derivation of the completion of K with respect to P, and we see immediately that it suffices to prove the theorem when K is complete. Since $Dk \subset k$, D maps any element of K that is algebraic over k into another element algebraic over k. Thus k may be replaced by its algebraic closure in K. By the lemma the residue field is now just k, so that if we choose $t \in K$ such that $\operatorname{ord}_P t = 1$ we have $K = k((t))$, the field of formal power series in t with coefficients in k. Write $x = \sum_{n \geqq i} a_n t^n$, $y = \sum_{n \geqq j} b_n t^n$, with each $a_n, b_n \in k$, $a_i \neq 0$, $b_j \neq 0$. Since $\{x(P), y(P)\} \subset \{0, \infty\}$ we have $\operatorname{ord}_P x = i \neq 0$, $\operatorname{ord}_P y = j \neq 0$. By the continuity of D we have $Dx = \sum_{n \geqq i} (Da_n) t^n + (\sum_{n \geqq i} n a_n t^{n-1}) Dt$ and a similar expression exists for Dy. We shall show that the two cases (1) and (2) occur in the cases $\operatorname{ord}_P Dt > 0$ and $\operatorname{ord}_P Dt \leqq 0$ respectively. If $\operatorname{ord}_P Dt > 0$ we clearly have $\operatorname{ord}_P Dx \geqq i$, $\operatorname{ord}_P Dy \geqq j$, so that $\operatorname{ord}_P(Dx/x), \operatorname{ord}_P(Dy/y) \geqq 0$. In particular, if $z \in K$ has positive order at P then so has Dz. If $z = \sum_{n \geqq 0} c_n t^n$, with each $c_n \in k$, then $Dz = Dc_0 + (\text{terms of order} > 0)$, so that $(Dz)(P) = Dc_0 = D(z(P))$. Thus everything has been verified for case (1). We now proceed to case (2), where we have $\operatorname{ord}_P Dt \leqq 0$. Let the leading term of Dt be ct^m, with $c \in k$, $c \neq 0$ and $m \leqq 0$. Then the leading terms of Dx and Dy are, respectively, $ia_i c t^{m+i-1}$ and $jb_j c t^{m+j-1}$. Thus $\operatorname{ord}_P(Dx/x) = \operatorname{ord}_P Dx - \operatorname{ord}_P x = m - 1 = \operatorname{ord}_P(Dy/y)$. Hence $\operatorname{ord}_P(y/x) = \operatorname{ord}_P(Dy/Dx)$. The last equation to be verified is now clear if $\operatorname{ord}_P(y/x)$ is nonzero, both sides then being either 0 or ∞. In the final case $\operatorname{ord}_P(y/x) = 0$ we get for the leading terms of y/x and of Dy/Dx, respectively, $a_i t^i/(b_j t^j) = a_i t^{i-j}/b_j$ and $ia_i c t^{m+i-1}/(jb_j c t^{m+j-1}) = ia_i t^{i-j}/(jb_j)$, and these are equal since now $i = j$.

We remark that in at least one case it is unnecessary to assume that D is continuous in the topology of P, since this will automatically be true, and that is the case where K is a finite extension of k of transcendence degree one [3, Lemma 1]. But in general the continuity assumption cannot be omitted, as the following example shows: k is any field of characteristic zero, $K = k(x, y)$, where x and y are algebraically independent over k, $D = \partial/\partial y$, the derivation on K which is zero on k and x and sends y into 1, and P is the k-place of K associated with the formal power series embedding $K \to k((t))$ given by $f(x, y) \mapsto f(t, e^t)$, where e^t is the usual series $\sum_{n \geqq 0}^{\infty} t^n/n!$. That D is not continuous may be seen by noting that for any integer $m \geqq 0$ we have

$$\operatorname{ord}_P\left(y - \sum_{n=0}^{m} x^n/n!\right) = m + 1$$

while

$$\operatorname{ord}_P D\left(y - \sum_{n=0}^{m} x^n/n!\right) = \operatorname{ord}_P Dy = \operatorname{ord}_P 1 = 0.$$

We proceed to give some applications of the theorem. Recall that a *liouvillian extension* of a differential field k is a differential extension field K of k which is obtained by repeated extensions by integrals, or exponentials of integrals, or algebraic elements, that is an extension of k of the form $k(t_1, t_2, \cdots, t_n)$, where for each $i = 1, \cdots, n$ either t_i' is in $k(t_1, \cdots, t_{i-1})$, or t_i'/t_i is in $k(t_1, t_2, \cdots, t_{i-1})$, or t_i is algebraic over $k(t_1, t_2, \cdots, t_{i-1})$.

PROPOSITION. *Let k be a differential field of characteristic zero, let n be a positive integer, and let f be a polynomial in several variables with coefficients in k and of total degree less than n. Then if the differential equation*

$$y^n = f(y, y', y'', \cdots)$$

has a solution in some liouvillian extension field of k, it has a solution in an algebraic extension field of k.

It clearly suffices to prove this under the simpler assumption that the equation has a solution y in a differential extension field K of k which is a finite algebraic extension of a field $k(t)$, where t is transcendental over k and either t' or t'/t is in k. Let P be a pole of t, that is, a k-place of K such that $\operatorname{ord}_P t < 0$. Then in either case $t' \in k$ or $t'/t \in k$ we get $\operatorname{ord}_P t'/t \geqq 0$. The derivation on K is continuous in the topology of P, by the remark immediately following the proof of the theorem. Hence case (1) holds, and for any $x \in K$ we have $\operatorname{ord}_P x' \geqq \min(0, \operatorname{ord}_P x)$. Thus $\operatorname{ord}_P y^{(m)} \geqq \min(0, \operatorname{ord}_P y)$ for all $m \geqq 0$. If $\operatorname{ord}_P y < 0$ then $\operatorname{ord}_P f(y, y', y'', \cdots) \geqq (n-1)\operatorname{ord}_P y > n \operatorname{ord}_P y = \operatorname{ord}_P y^n$, a contradiction. Therefore $\operatorname{ord}_P y^{(m)} \geqq 0$ for all $m \geqq 0$. Hence each $y^{(m)}(P)$ is finite, therefore algebraic over k, with $y^{(m+1)}(P) = (y^{(m)}(P))'$, so that $y(P)$ is a solution of our differential equation that is algebraic over k.

As an application of the proposition, consider a differential field k of characteristic zero, elements $a_1, \cdots, a_n \in k$, and the homogeneous linear differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0.$$

If there is a nonzero solution y of this differential equation and we set $u=y'/y$ then we get $y'=uy$, $y''=u'y+uy'=(u'+u^2)y$, $y^{(3)}=(u''+2uu')y+(u'+u^2)y'=(u''+3u'u+u^3)y, \cdots, y^{(n)}=(u^{(n-1)}+nu^{(n-2)}u+\cdots+u^n)y$, and our differential equation becomes

$$u^{(n-1)} + nu^{(n-2)}u + \cdots + u^n + a_1(u^{(n-2)} + \cdots + u^{(n-1)}) + \cdots + a_n = 0.$$

Conversely, if the element u of a differential extension field of k satisfies this last equation then an element y of a larger differential extension field of k such that $y'/y=u$ will be a nonzero solution of our original homogeneous linear differential equation. A nonzero solution y of the homogeneous linear differential equation will be an element of a liouvillian extension field of k if and only if the corresponding solution u of the nonlinear differential equation is an element of a liouvillian extension field of k. But the nonlinear differential equation is of the type considered in the proposition, so that if it has a solution in a liouvillian extension of k then it has a solution in an algebraic extension of k. Hence *if the homogeneous linear differential equation has a nonzero solution in a liouvillian extension of k then it has a nonzero solution y such that y'/y is algebraic over k.* Kolchin has given a proof of this (unpublished) by means of his Picard-Vessiot theory. The case $n=2$ goes back to Liouville (cf. [2, p. 70]; Ritt himself gives a formal power series proof); in this case the nonlinear differential equation is the Riccati equation $u'+a_1u+a_2+u^2=0$ associated with the given homogeneous linear differential equation.

We now show how the proposition may be used to prove that elliptic functions are not liouvillian. To do this it suffices to show that any element y of a liouvillian extension of the field $C(z)$ of rational functions of the complex variable z which satisfies a differential equation

$$(y')^2 = y^3 + ay + b, \quad a, b \in C, \qquad a^3/27 + b^2/4 \neq 0,$$

is necessarily constant. Note that all elements of C are constant and that $C(z)$ is itself a liouvillian extension of C, since $z'=1$. As above, it remains only to show that if $k \subset K$ are differential extension fields of C, with K a finite algebraic extension of $k(t)$, where t is transcendental over k and either $t' \in k$ or $t'/t \in k$, and if there exists a nonconstant solution y of the above differential equation in K, then there exists a nonconstant solution that is algebraic over k. Again let the k-place P of K be any pole of t. The proposition shows that $y(P)$ is a solution of the differential equation, and we are all done unless it happens that $y'(P)=0$. In this troublesome case, since we have $y'\neq 0$, the place P induces a nontrivial C-place of the elliptic function field over C given by $C(y, y')=C\langle y\rangle$, which is the function field over C of the cubic curve $Y_2^2= Y_1^3+aY_1+b$. The points of this curve that are rational over C have a well-known commutative group structure.

Each point of the curve that is rational over C produces a translation of the points of the curve which is an automorphism of the curve, equivalent to one of its function field $C(y, y')$, and by Kolchin's galois theory [**1**, p. 807] the latter is a differential automorphism. For only a finite number of such differential automorphisms σ of $C\langle y\rangle$ does the place induced by P on the cubic curve go into one of the finite number of zeros of y', so for any σ distinct from a finite set we have $(\sigma y')(P)\neq 0$. In this case $(\sigma y)(P)$ is a nonconstant solution of the differential equation that is algebraic over k, and the proof is complete. We remark that in [**2**, p. 87] Ritt proves in an entirely different manner the weaker statement that elliptic functions are not elementary. We also remark that the nonliouvillian character of elliptic functions, indeed the fact that they cannot be obtained by repeated Picard-Vessiot extensions of $C(z)$, can be proved directly by Kolchin's methods, based on the idea that any homomorphism between connected algebraic groups, one linear, the other an abelian variety, is trivial.

References

1. E. Kolchin, *Galois theory of differential fields*, Amer. J. Math. **75** (1953), 753–824. MR **15**, 394; 1140.

2. J. F. Ritt, *Integration in finite terms*, Columbia Univ. Press, New York, 1948. MR **9**, 573.

3. M. Rosenlicht, *On the explicit solvability of certain transcendental equations*, Inst. Hautes Études Sci. Publ. Math. No. 36 (1969), 15–22. MR **41** #3454.

Department of Mathematics, University of California, Berkeley, California 94720