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WEYL'S LEMMA FOR POINTWISE SOLUTIONS OF ELLIPTIC EQUATIONS

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ABSTRACT. We prove that pointwise, L_1 solutions of second order elliptic partial differential equations are classical solutions.

0. **Introduction.** A consequence of Weyl's lemma for second order elliptic partial differential equations is that every L_1 (Lebesgue class) weak solution is a classical solution, i.e., if, for

$$L = \sum_{i,j=1}^{K} a_{ij}(x)\,\partial^2/\partial x_i\,\partial x_j + \sum_{i=1}^{K} b_i(x)\,\partial/\partial x_i + c(x) \tag{1}$$

and

$$L^* = \sum_{i,j=1}^{K} \partial^2/\partial x_i\,\partial x_j a_{ij}(x) - \sum_{i=1}^{K} \partial/\partial x_i b_i(x) + c(x),$$

we have $\int_\Omega uL^*\phi = \int_\Omega f\phi$ for all ϕ in $C_c^\infty(\Omega)$, then $Lu=f$ in Ω.

The differentiability of the coefficients of L, required for the definition of L^*, is not intrinsic in view of the maximum principle and the Dirichlet problem (in the case $c(x)\leqq 0$). Our aim is to employ a notion of generalized solution of $Lu=f$ which bypasses the adjoint operator and thereby establish an analogue of Weyl's lemma when the coefficients are in a Hölder-α class. Definitions and the statement of the theorem follow in the next section.

1. **Preliminaries.** We shall work in K-dimensional Euclidean space, R^K, $3\leqq K$, and shall use the following notation: $x=(x_1, \cdots, x_K)$ and $B(x, r)=$the open K-ball centered at x with radius r. Ω will denote an open set in R^K; $|E|$, the Lebesgue measure of E; ∂E, the boundary of E; subscripted e, constants which depend only on the operator (1) and K; and ω_K, the surface area of $\partial B(0, 1)$.

A function u, which is in L_1 in a neighborhood of x, is said to be in

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$t_\beta(x)$ if there is a polynomial $P(y)$ of degree $\leqq\beta$ such that

$$(2) \qquad r^{-K}\int_{B(0,r)} |u(x+y)-P(y)|\,dy = o(r^\beta) \quad \text{as } r\to 0.$$

For $\beta=2$, $P(y)=\gamma_0+\sum_{i=1}^K \gamma_i y_i+2^{-1}\sum_{i,j=1}^K \gamma_{ij}y_iy_j$, where $\gamma_{ij}=\gamma_{ji}$; we define the generalized partial derivatives $D_iu(x)$, $D_{ij}u(x)$ to be γ_i, γ_{ij} respectively. Also we can redefine u at x, if necessary, so that $u(x)=\gamma_0$. (The class t_β was defined in **[1]** and has since appeared in **[2]** and **[6]**.)

For u in $t_2(x)$, we say that u is a t_2 solution of $Lu=f$ at x if

$$(3) \qquad \sum_{i,j=1}^K a_{ij}(x)D_{ij}u(x) + \sum_{i=1}^K b_i(x)D_iu(x) + c(x)u(x) = f(x).$$

We say that u is a t_2 solution of $Lu=f$ in Ω if u is a t_2 solution at each point in Ω. u is said to be a classical solution if $u\in C^2(\Omega)$. (It should be noted that no restrictions have been placed on the operator L up to this point.)

We shall assume the operator (1) to be elliptic and have α-Hölder continuous coefficients, $0<\alpha\leqq 1$; with this, (1) is uniformly elliptic on compact subsets of Ω.

THEOREM. *If u is an L_1, t_2 solution of $Lu=f$ in Ω, then u is a classical solution.*

REMARKS. No assumption has been made on the integrability of the generalized derivatives $D_iu(x)$, $D_{ij}u(x)$ when considered as functions on Ω. In fact, see **[2]**, there are simple examples of functions in $t_2(x)$, x in $(-1, 1)$ such that $D_1u(x)$, $D_{11}u(x)$ are not in L_1 on compact subsets of $(-1, 1)$; note that the $o(r^\beta)$ in (2) is not assumed to be uniform in x.

The special case of u in L_∞, $f\equiv 0$, and $c(x)\leqq 0$, has been established in **[2]**. The case for general f, α-Hölder continuous, is immediate. The restriction on $c(x)$ is not essential and while a nontrivial argument is necessary for unrestricted $c(x)$, the techniques are essentially those of [**2**, §§1–4], and those of this paper. We therefore maintain $c(x)\leqq 0$.

As in **[2]**, we only need assume that u is a t_2 solution of $Lu=f$ almost everywhere and in t_2 everywhere in Ω. We shall omit this extension.

It is not clear that the Hölder-α condition is best possible. However, our result is best possible with respect to the notion of t_2 solution in that the conclusion fails if the absolute values in (2) are not required, i.e., $u(x)=x_1|x|^{-K}$ satisfies, with $P(y)\equiv 0$ for $x=0$,

$$r^{-K}\int_{B(0,r)} u(x+y)-P(y)\,dy = o(r^2) \quad \text{as } r\to 0$$

for all x; that is, $x_1|x|^{-K}$ has generalized Laplacian equal to zero everywhere and is clearly not harmonic at 0.

Our theorem combined with [2] allows for an immediate generalization of Serrin's Theorem 1 [4] with respect to removable F_σ sets.

A forthcoming paper in the Ann. Scuola Norm. Sup. Pisa by Hager and Ross deals with elliptic equations in divergence form with α-Hölder coefficients; the notion of weak solution is considered vis-à-vis the notion of pointwise solution employed in this paper.

This work is an extension of some of the concepts developed by V. L. Shapiro in [6].

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2. **Fundamental lemmas.** The first three lemmas were established in [2]. By a portion P of a set Z, we shall mean a nonempty intersection of Z with an open ball.

LEMMA 1 [2, LEMMA 2]. *If u is in $t_2(x)$ for all x in Z and if Z is closed and nonempty, then there is a portion P of Z such that u is continuous in P in the relative topology, i.e., if $x_1 \in P$, then given $\varepsilon>0$, there is a $\delta>0$ such that for $x_2 \in P$ and $|x_1-x_2|<\delta$, $|u(x_1)-u(x_2)|<\varepsilon$.*

LEMMA 2 (SEE [2, PROOF OF THEOREM 1]). *If u is in $t_2(x)$ for all x in Z, closed and nonempty, then there is a portion P of Z and positive constants M and r_0 such that for all x in P and $0<r<r_0$,*

$$|B(0, r)|^{-1}\int_{B(0,r)} |u(x+y)-u(x)|\, dy \leqq M. \tag{4}$$

LEMMA 3 [2, THEOREM 1]. *If u is an L_∞, t_2 solution of $Lu=0$ $(c(x)\leqq 0)$ in Ω, then u is a classical solution.*

LEMMA 4 (J. SERRIN, [3, p. 300]). *There exist functions $K_+(x, y)$ and $K_-(x, y)$ for $|x|\leqq r$, $|y|=r$, $x\neq y$, and $r\leqq 1$ having the following properties:*

(i) *Considered as functions of x for fixed y,*

$$LK_+ \geqq 0 \quad \text{and} \quad LK_- \leqq 0. \tag{5}$$

(ii) *For any point y_0, $|y_0|=r$, and any continuous function $g(y)$,*

$$\lim_{x\to y_0}\int_{\partial B(0,r)} K_\pm(x, y)g(y)\, dS_r(y) = g(y_0) \tag{6}$$

where $dS_r(y)$ is the natural surface area element.

(iii) *There are positive constants e_1 and e_2 such that, for $|x|\leqq r/3$,*

$$K_+(x, y) \geqq e_1 r^{1-K} \quad \text{and} \quad K_-(x, y) \leqq e_2 r^{1-K}. \tag{7}$$

LEMMA 5. *If $Lu=0$ in $B(x_0, r)$, then*

$$|u(x_0)| \leqq e_3\, |B(x_0, r)|^{-1}\int_{B(x_0,r)} |u(x)|\, dx. \tag{8}$$

PROOF. As in [4], $u(x)=\int_{\partial B(x_0,\rho)} u(y)\, d\omega(x,y)$ where $d\omega(x,y)$ is a nonnegative Borel measure on $\partial B(x_0, \rho)$, with $\rho<r$. Thus

$$|u(x)| \leqq \int_{\partial B(x_0,\rho)} |u(y)|\, d\omega(x, y).$$

Form

$$u_-(x) = \int_{\partial B(x_0,\rho)} K_-(x, y)\, |u(y)|\, dS_\rho(y).$$

By (5), (6), and the maximum principle,

$$\int_{\partial B(x_0,\rho)} |u(y)|\, d\omega(x, y) \leqq u_-(x) \quad \text{for } x \text{ in } B(x_0, \rho),$$

and by (7),

$$u_-(x) \leqq e_2 \rho^{1-K} \int_{\partial B(x_0,\rho)} |u(y)|\, dS_\rho(y) \quad \text{for } |x - x_0| \leqq \rho/3.$$

Hence

$$|u(x_0)| \leqq e_2 \rho^{1-K} \int_{\partial B(x_0,\rho)} |u(y)|\, dS_\rho(y).$$

Thus by integrating

$$\frac{r^K}{K} |u(x_0)| \leqq e_2 \int_{B(x_0,r)} |u(y)|\, dy$$

which yields (8).

PROOF OF THE THEOREM ($c(x)\leqq 0$). Let Z be the set of discontinuities of u; $\bar{Z}$ is its closure. If $\bar{Z}$ is empty, then by Lemma 3, u is a classical solution. Assuming, therefore, that $\bar{Z}\neq\varnothing$, we have, by Lemmas 1 and 2, a portion P of $\bar{Z}$ in which u is continuous in the relative topology and satisfies (4) for x in P. Let $x_0 \in P$. If we can show that there is a $\delta>0$ such that u is bounded in $|x-x_0|<\delta$, then u is a classical solution in $|x-x_0|<\delta$. Consequently x_0 is not in $\bar{Z}$; therefore x_0 is not in P, which is a contradiction based on the assumption that $\bar{Z}\neq\varnothing$.

We can choose $\delta=\min(r_0/2, r_1/2)$, where r_0 is given in Lemma 2 and where r_1 is selected so that

(i) $0<r_1$,

(ii) $\{|x_0-x|<r_1, x \text{ in } \bar{Z}, |u(x)-u(x_0)|<1\}\subseteq P$.

For $|x-x_0|<\delta$, x in $\bar{Z}$,

$$|u(x)| \leqq |u(x) - u(x_0)| + |u(x_0)| \leqq |u(x_0)| + 1.$$

For $|x-x_0|<\delta$, x not in $\bar{Z}$, there is a point x^* in P and $0<\rho<\delta$ such that $x^* \in \partial B(x, \rho)$ while $P\cap B(x, \rho)=\varnothing$. Thus $Lu=0$ in $B(x, \rho)$. Hence, by (8),

$$|u(x)| \leqq e_3\, |B(x, \rho)|^{-1} \int_{B(x,\rho)} |u(y)|\, dy$$
$$\leqq e_3 2^K\, |B(x^*, 2\rho)|^{-1} \int_{B(x^*,2\rho)} |u(y)|\, dy.$$

Since $\rho<\delta<r_0/2$, $2\rho<r_0$, and $x^* \in P$,

$$\leqq e_3 2^K[|u(x^*)| + M].$$

Since $|x_0-x^*|\leqq|x_0-x|+|x-x^*|\leqq\delta+\rho<r_1$, $|u(x^*)|\leqq|u(x_0)|+1$, which gives $|u(x)|\leqq e_3 2^K[|u(x_0)|+M+1]$, completing the proof.

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