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RESOLUTIONS AND PERIODICITY

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ABSTRACT. The representation theory of groups with a cyclic Sylow subgroup is used to give a detailed construction of minimal projective resolutions.

The Artin-Tate theorem [1, p. 262] has as a consequence that a Sylow p-subgroup of a finite group G, for an odd prime p, is cyclic exactly when the cohomology $H^n(G, Z_p)$ is periodic for positive integers n. Moreover, in this case Swan [4] has shown that the minimal projective resolution of Z_p over $Z_p[G]$ is periodic. By using the deep theory of groups with a cyclic Sylow subgroup we can go one step further.

THEOREM. *If G is a finite group with a cyclic Sylow p-subgroup for a prime p and F is a splitting field of characteristic p for G then each term of the minimal projective resolution of F over $F[G]$ is indecomposable.*

Actually, we shall prove much more, but the indecomposability seems to be the most important conclusion. To describe our results fully we need some notation. Let $e=|N(S_p):C(S_p)|$, where S_p is a Sylow p-subgroup of G, and $V_1=F$, $V_2, \cdots, V_e$ be irreducible $F[G]$ modules in the principal p-block B_0 of G, one from each isomorphism class of such modules. Let U_i be an indecomposable projective $F[G]$ module which has V_i as a homomorphic image so that U_i is determined to within isomorphism. Finally, let T be the Brauer tree corresponding to B_0 so that the vertices are in one-to-one correspondence with the p-conjugacy classes of ordinary irreducible characters in B_0 while the edges correspond one-to-one with the isomorphism classes of irreducible $F[G]$ modules in B_0. Hence, if E is an edge, let $V_E=V_i$ when the isomorphism class of V_i corresponds to E. In this case, we also set $U_E=U_i$, the corresponding projective. Each vertex P of T is assigned a multiplicity $m(P)$; it is known that $m(P)>1$ for at most one vertex P. Moreover, there is a canonical circular ordering of the edges emanating from P determined by the structure of the

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indecomposable projective $F[G]$ modules in the following way. The projective U_E has a unique maximal submodule M_E which is in turn a sum

$$M_E = K_E(P) + K_E(Q),$$

in which $K_E(P)$ and $K_E(Q)$ are uniserial modules and P and Q are the vertices on $E=(PQ)$. The module $K_E(P)$ has a unique composition series and the composition factors (from the "top down") are $V_{E_1}, V_{E_2}, \cdots, V_E$, where $E_1, E_2, \cdots, E$ are all the edges on P and the initial sequence of distinct V's is repeated $m(P)$ times. A similar statement holds for the vertex Q. The circular ordering on the edges of P is determined by the above sequence in which E_1 follows E. The intersection $K_E(P) \cap K_E(Q) = S(E)$ is the unique minimal submodule of U_E and is isomorphic with V_E.

We shall now describe a canonical infinite sequence of edges of T. Let E_0 be the unique edge connected to the vertex P_0 corresponding to the principal character and let P_1 be the other vertex connected to E_0. Suppose now that we have defined edges $E_0, \cdots, E_n$ and vertices $P_0, P_1, \cdots, P_{n+1}$; let E_{n+1} be the edge immediately following E_n in the circular ordering around P_{n+1} and let P_{n+2} be the other vertex connected to E_{n+1}. Since T is a tree with exactly e edges it follows readily that in the sequence $E_0, \cdots, E_{2e-1}$ each edge appears exactly twice and that $E_{i+2e}=E_i$ for all i.

THEOREM 1. *If G is a finite group with cyclic Sylow p-subgroup, F is a splitting field of characteristic p for G and*

$$\to X_{n+1} \to X_n \to \cdots \to X_1 \to X_0 \to F \to 0$$

is the minimal projective resolution for F over $F[G]$, then $X_i \simeq U_{E_i}$, $i>0$, where E_i is the ith edge in the canonical sequence of edges of the Brauer tree for the principal p-block of G.

In view of the remarks above, the nature of the period $2e$ is "explained". A similar result, in which certain p-adic rings of characteristic zero replace F, is now immediate.

As a first step in the rather short proof we establish the following

LEMMA. *Let P be a vertex of T, E and F consecutive edges on P and let Q be the other vertex connected to F. There exists an exact sequence*

$$0 \to K_F(Q) \to U_F \to K_E(P) \to 0.$$

PROOF. The module $K_E(P)$ is uniserial with "top" composition factor isomorphic to V_F, since F follows E in the ordering around P. Thus, $K_E(P)$ is a homomorphic image of U_F. The kernel of this map contains the socle $S(F)$ of U_F since $K_E(P)$ is not projective and so is not isomorphic to U_F. Moreover, this kernel must contain $K_F(Q)$ because none of the composition factors of $K_F(Q)/S(F)$ are composition factors of $K_E(P)$. The composition lengths of $U_F/K_F(Q)$ and $K_E(P)$ are each equal to the product of $m(P)$ and the number of edges on P, so the lemma follows.

PROOF (OF THEOREM 1). Let $E_0=(P_0P_1)$, $E_1=(P_1P_2), \cdots$ be the canonical sequence of edges. Iterated application of the lemma yields the exact sequence

$$\cdots \to U_{E_2} \to U_{E_1} \to U_{E_0} \to K_{E_0}(P_0) \to 0,$$

in which the image of $U_{E_{n+1}}$ in U_{E_n} is $K_{E_n}(P_n)$. But $m(P_0)=1$ and E_0 is the unique edge connected to P_0 so $K_{E_0}(P_0)$ is the minimal submodule of U_{E_0} and $K_E(P_0)\simeq F$. Hence, we have obtained a resolution for F, which is clearly minimal since each term is indecomposable.

Minimal resolutions of certain other irreducible modules can be similarly described. If the module V_E corresponds to the edge $E=(PQ)$ with E the only edge connected to P and $m(P)=1$ then the theorem holds as stated for the minimal resolution of V_E using a canonical sequence starting with $E_0=E$ and $P_0=P$. Again the period is $2e$. It is known that E satisfies these conditions if and only if V_E is the reduction modulo p of an $R[G]$ module for a suitable p-adic ring R of characteristic zero (see Corollary 7.3 of [2]). When E is an edge not satisfying these conditions, the minimal projective resolution of V_E will always contain decomposable terms and appears to be considerably more difficult to describe.

For example, let $G=\mathrm{PSL}(2,p)$ with p a prime exceeding three. In this case $e=\frac{1}{2}(p-1)$ and the Brauer tree is an open polygon with edges

$$E_0 = (P_0P_1), E_1 = (P_1P_2), \cdots, E_{e-1} = (P_{e-1}P_e)$$

and where $m(P_e)=2$. The minimal projective resolution for $V_{E_{e-1}}$ contains a term which is the sum of e indecomposable projectives. However, this is as "bad" as the situation can get, as we shall now show.

THEOREM 2. *Let G and F be as in Theorem 1 and let M be any indecomposable $F[G]$ module. If the minimal projective resolution for M over $F[G]$ is*

$$\to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$$

then X_i, $i \geqq 0$, has no repeated indecomposable summand. In particular, X_i is the sum of at most e indecomposable $F[G]$ modules where e is the number of irreducible $F[G]$ modules in the p-block containing M.

PROOF. Let $X_{-1}=M$ and let D_i, $i\geqq 0$, be the kernel of the map from X_i to X_{i-1}. Since M is indecomposable and the resolution is minimal, it follows that D_0 is indecomposable [3]. Iterating this argument we deduce that each D_i is indecomposable, $i\geqq 0$. By Corollary 7.7 of [2] the socle of D_i is multiplicity free. The minimality of the resolution guarantees that the socle of D_i and the socle of X_i coincide. The theorem follows.

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